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Path-Connectivity of Fréchet Spaces of Graphs

Erin Chambers
St Louis University, USA

Brittany Terese Fasy
Montana State University, USA

Benjamin Holmgren
Montana State University, USA

Sushovan Majhi
University of California - Berkeley, USA

Carola Wenk
Tulane University, USA

Abstract

We examine topological properties of spaces of paths and graphs mapped to $\mathbb{R}^d$ under the Fréchet distance. We show that the spaces of graphs and paths mapped to $\mathbb{R}^d$ are path-connected if the map is either continuous or an immersion. If the map is an embedding, we show that the space of paths is path-connected, while the space of graphs only maintains this property in dimension 4 or higher.

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1 Introduction

Motivated by the ubiquitous nature of one-dimensional data in a Euclidean ambient space (road networks in $\mathbb{R}^2$, for example), we investigate spaces of paths and graphs in $\mathbb{R}^d$. In particular, we examine these spaces in relation to the Fréchet distance, which is widely studied in the computational geometry literature [1–3, 5–7]. We work with three classes of paths: the set $\Pi_\mathcal{C}$ of all paths continuously mapped into $\mathbb{R}^d$, the set $\Pi_\mathcal{E}$ of all paths embedded in $\mathbb{R}^d$, and the set $\Pi_\mathcal{I}$ of all paths immersed in $\mathbb{R}^d$. In addition, we study three analogous spaces of graphs: the set $\mathcal{G}_\mathcal{C}$ of all graphs continuously mapped into $\mathbb{R}^d$, the set $\mathcal{G}_\mathcal{E}$ of all graphs embedded in $\mathbb{R}^d$ and the set $\mathcal{G}_\mathcal{I}$ of all graphs immersed in $\mathbb{R}^d$. See Figure 1 for examples of paths in $\mathbb{R}^2$. We then topologize these sets using the open ball topology under the Fréchet distance, and study their path-connectedness property.

2 Background

We define the Fréchet distance for graphs, inspired by the Fréchet distance among paths [1]. Let G be an abstract graph, and let $\phi, \psi\colon G \to \mathbb{R}^d$ be continuous, rectifiable maps. Given any homeomorphism $h\colon G \to G$, we say that the *induced* L_∞ *distance* between the maps ϕ

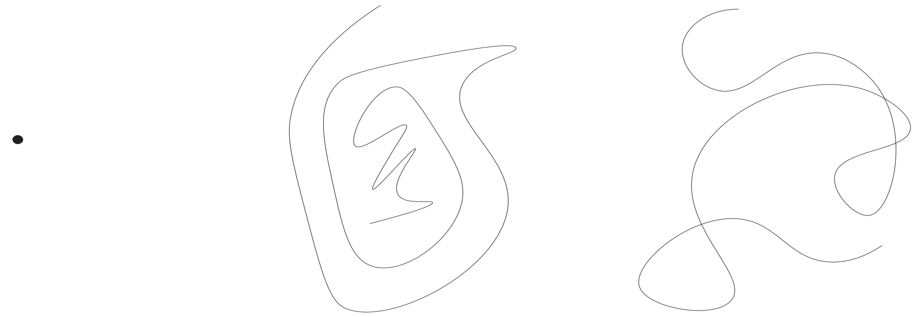


Figure 1 The images of an element in $\Pi_{\mathcal{C}}$, $\Pi_{\mathcal{E}}$, and $\Pi_{\mathcal{I}}$ respectively, mapped in $\mathbb{R}^2$.

and $\psi \circ h$ is $||\phi - \psi \circ h||_\infty = \max_{x \in G} |\phi(x) - \psi(h(x))|$. With this distance in hand, we define the Fréchet distance between (G, ϕ) and (G, ψ) by minimizing over all homeomorphisms: [1]

$$d_{FG}\left((G,\phi),(G,\psi)\right) := \min_h ||\phi - \psi \circ h||_\infty$$

We now define and provide context for the underlying spaces that are studied in this work. Recall from above that $\Pi_{\mathcal{C}}$ denotes the set of all continuous mappings $\alpha : [0,1] \to \mathbb{R}^d$. The set $\Pi_{\mathcal{E}}$ of embedded paths in $\mathbb{R}^d$ results from further specifying that α is injective, and the set $\Pi_{\mathcal{I}}$ of immersed paths in $\mathbb{R}^d$ results from requiring only local injectivity of α. Note that $\Pi_{\mathcal{E}} \subsetneq \Pi_{\mathcal{I}} \subsetneq \Pi_{\mathcal{C}}$ and elements of $\Pi_{\mathcal{C}}, \Pi_{\mathcal{E}}$, and $\Pi_{\mathcal{I}}$ are deemed equivalent if the image of their underlying map α is equivalent, giving a path-Fréchet distance (denoted d_{FP}) of zero.

We define the analogous spaces of graphs, letting G be an abstract graph and $\mathcal{G}_{\mathcal{C}}(G)$ denote the set of all continous mappings $\phi : G \to \mathbb{R}^d$. Similarly, we define the set of embeddings $\mathcal{G}_{\mathcal{E}}(G)$ with the added requirement that ϕ be injective, and the set of immersions $\mathcal{G}_{\mathcal{I}}$ with the requirement that ϕ need be only locally injective. Note that elements of $\mathcal{G}_{\mathcal{C}}, \mathcal{G}_{\mathcal{I}}$, and $\mathcal{G}_{\mathcal{E}}$ are equivalent (with graph Fréchet distance zero) if their underlying graphs belong to the same homeomorphism class, and if the image of their accompanying map ϕ is equivalent.

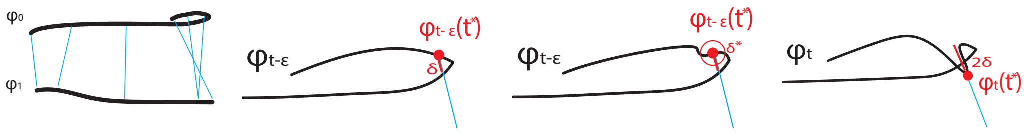
3 Results

▶ **Theorem 1** (Continuous Mappings). *The topological spaces of continuous mappings of paths $(\Pi_{\mathcal{C}}, d_{FP})$ and continuous mappings of graphs $(\mathcal{G}_{\mathcal{C}}(G), d_{FG})$ in $\mathbb{R}^d$ are path-connected.*

Proof Sketch. Let $\phi_0, \phi_1 \in \Pi_{\mathcal{C}}$. Naively, a path may be constructed from ϕ_0 to ϕ_1 by interpolating ϕ_0 to ϕ_1 along the pointwise matchings (so-called leashes) defining $d_{FP}(\phi_0, \phi_1)$. The same technique may be extended to demonstrate the path-connectivity of $\mathcal{G}_{\mathcal{C}}(G)$. ◀

▶ **Theorem 2** (Immersions). *The topological spaces of immersions of paths $(\Pi_{\mathcal{I}}, d_{FP})$ and immersions of graphs $(\mathcal{G}_{\mathcal{I}}(G), d_{FG})$ in $\mathbb{R}^d$ are path-connected.*

[1] Other generalizations of the Fréchet distance minimize over all "orientation-preserving" homeomorphisms, which can be defined in several ways for stratified spaces. We drop this requirement in our definition.



(a) Interpolate **(b)** δ from crossing **(c)** Inflate δ^*-nbhd **(d)** Self-cross by 2δ

Figure 2 The sequence of moves to continuously conduct self crossings in $\Pi_{\mathcal{I}}$.

Proof Sketch. Let $\phi_0, \phi_1 \in \Pi_{\mathcal{I}}$, and construct a path $\Gamma : [0,1] \to \Pi_{\mathcal{I}}$ as in Theorem 1 by interpolating ϕ_0 to ϕ_1 along the pointwise matchings defining $d_{FP}(\phi_0, \phi_1)$. At some $t \in [0,1]$, $\phi_t = \Gamma(t)$ could create an intersection not present in ϕ_0. This may collapse an entire region of the image of ϕ_t, rendering ϕ_t no longer an immersion. Then, there exists $\epsilon > 0$ such that $\Gamma(t-\epsilon) = \phi_{t-\epsilon}$ has $t^* \in [0,1]$ where $\phi_{t-\epsilon}(t^*)$ is $\delta > 0$ away from a new self-intersection, and t^* comes sufficiently close to minimizing δ. At this time $t - \epsilon$, suspend interpolation along all leashes, and continuously inflate a small δ^*-neighborhood $\phi_{t-\epsilon}|_{(t^*-\delta^*, t^*+\delta^*)}$ about the point $\phi_{t-\epsilon}(t^*)$ in the image of $\phi_{t-\epsilon}$ so that the leash lengths for every point in the δ^*-neighborhood equal the leash length defined at $\phi_{t-\epsilon}(t^*)$. Then directly perturb $\phi_{t-\epsilon}(t^*)$ by 2δ along its unique leash such that the crossing at $\phi_{t-\epsilon}(t^*)$ occurs, and the crossing point defined by t^* again lies δ away from a self intersection, and 2δ away from its original position in the final image of ϕ_t. See Figure 2d. Repeat the process for any subsequent crossings in the interpolation. An analogous path can be constructed for graphs. ◀

▶ **Theorem 3** (Path Embeddings). *The space* $(\Pi_{\mathcal{E}}, d_{FP})$ *is path-connected.*

Proof Sketch. Let $\phi_0, \phi_1 \in \Pi_{\mathcal{E}}$. There exists a canonical path from ϕ_0 to ϕ_1 by condensing each map toward its center until the images are "nearly straight", continuously mapping each image to a straight segment, and then interpolating as in Theorem 1. ◀

▶ **Theorem 4** (Graph Embeddings). *The topological space of graphs* $(\mathcal{G}_{\mathcal{E}}(G), d_{FG})$ *embedded in* $\mathbb{R}^d$ *is path-connected if* $d \geq 4$.

Proof Sketch. Examining the path-connectivity of $\mathcal{G}_{\mathcal{E}}$ under the Fréchet distance reduces to a knot theory problem for $d \leq 3$. For $d \geq 4$, there exists a sequence of Reidemeister moves from any tame knot to another. Hence, if $\phi_0, \phi_1 \in \mathcal{G}_{\mathcal{E}}$, we construct a path by interpolating along the pointwise matchings between ϕ_0 and ϕ_1 as in Theorem 1. If a self intersection would be created, we suspend interpolation elsewhere and conduct the corresponding Reidemeister move. Repeat the process for all intersections thereafter, until attaining the image of ϕ_1. ◀

▶ **Corollary 5** (Path-Connectivity of Metric Balls). *Metric balls in the space* $\Pi_{\mathcal{C}}, \mathcal{G}_{\mathcal{C}}(G), \Pi_{\mathcal{I}}$, *and* $\mathcal{G}_{\mathcal{I}}(G)$ *are path-connected.*

Proof Sketch. Note that the techniques used in Theorem 1 and Theorem 2 never strictly increase the Frechet distance among two images of corresponding maps, so metric balls in each space are path-connected. For Theorem 2 this relies on the inflation step in Figure 2c, which assures that the Fréchet distance is fixed during a crossing event. The paths constructed in Theorem 3 and Theorem 4 do not necessarily maintain this property. ◀

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A Additional Definitions Adapted from Topology and Geometry

▶ **Definition 6.** *The Open Ball Topology: Let $\mathbb{X}$ be a set and $d\colon \mathbb{X}\times\mathbb{X}\to\overline{\mathbb{R}}_{\geq 0}$ a distance (d need not be a metric). For each $r \geq 0$ and $x \in \mathbb{X}$, let $\mathbb{B}_d(x, r) := \{y \in \mathbb{X} \mid d(x, y) \leq r\}$; in words, $\mathbb{B}_d(x, r)$ denotes the open ball of radius r centered at x with respect to distance d. We use these open balls to generate a topology on $\mathbb{X}$, allowing x to range over $\mathbb{X}$ and r to range over all positive real numbers.*

▶ **Definition 7.** *Path-Connectivity: A topological space $\mathbb{X}$ is called* path-connected *if for any $a, b \in \mathbb{X}$, there exists a continuous map $\Gamma\colon [0, 1] \to \mathbb{X}$ joining a and b, i.e., $\Gamma(0) = a$ and $\Gamma(1) = b$. In this article, our attention is restricted to the Euclidean ambient space, so $X = \mathbb{R}^d$.*

▶ **Definition 8.** *The Fréchet Distance for Paths: Any continuous map $\alpha\colon [0, 1] \to \mathbb{R}^d$ is called a* path *in $\mathbb{R}^d$. Let $\Pi_\mathcal{C}$ denote the set of all paths in $\mathbb{R}^d$. Then, the* Fréchet distance *$d_{FP}\colon \Pi_\mathcal{C}\times\Pi_\mathcal{C}\to\overline{\mathbb{R}}_{\geq 0}$ between $\alpha_1, \alpha_2 \in \Pi_\mathcal{C}$ is defined as:*

$$d_{FP}(\alpha_1, \alpha_2) := \min_{r\colon [0,1]\to[0,1]} \max_{t\in[0,1]} |\alpha_1(t) - \alpha_2(r(t))|$$

where r ranges over all reparameterizations of the unit interval (that is, homeomorphisms such that $r(0) = 0$ and $r(1) = 1$), and $|\cdot|$ denotes the standard Euclidean norm.

▶ Remark 9. If $G = I$, then the relationship between the Fréchet distance between two paths $\alpha, \beta\colon I \to \mathbb{R}^d$ and the corresponding graphs $(I, \alpha), (I, \beta)$ is as follows:

$$d_{FG}\left((I, \alpha), (I, \beta)\right) = \min\left\{d_{FP}(\alpha, \beta), d_{FP}(\alpha, \beta^{-1})\right\},$$

where $\beta^{-1}\colon I \to \mathbb{R}^d$ is defined by $\beta^{-1}(t) = \beta(1 - t)$.

▶ **Definition 10** (Linear Combination of Graphs). *Let G denote an abstract graph. Let $\phi_0\colon G \to \mathbb{R}^d$ and $\phi_1\colon G \to \mathbb{R}^d$ be continuous, and let $\mathbf{G}_0 = (G, \phi_0)$ and $\mathbf{G}_1 = (G, \phi_1)$. If $h\colon G \to G$ is a homeomorphism and $c_0, c_1 \in \mathbb{R}$, then the linear combination $c_0\mathbf{G}_0 + c_1\mathbf{G}_1$ with respect to h is defined as follows: we define $\phi\colon G \to \mathbb{R}^d$ by $\phi(x) := c_0\phi_0(x) + c_1\phi_1(x)$. In short, we write $c_0\mathbf{G}_0 + c_1\mathbf{G}_1 = (G, \phi)$.*

Above, we observe that ϕ is continuous (since ϕ_0 and ϕ_1 are continuous). In addition, we note that linear combinations of graphs are defined on the specific representations *of the continuously mapped graphs, not* on the elements of $\mathcal{G}_\mathcal{C}$*. It is possible that two graphs are homemorphic $\mathbf{G}_0 \cong \mathbf{G}_0'$, but the corresponding linear combinations are not $(c_0\phi_0 + c_1\phi_1) \ncong (c_0\phi_0' + c_1\phi_1)$.*

B Spaces of Continuous Maps in $\mathbb{R}^d$

The proof sketch outlined in Theorem 1 is sufficient to demonstrate the path-connectivity of $\Pi_\mathcal{C}$, and can be extended to demonstrate the path-connectivity of $\mathcal{G}_\mathcal{C}$. The following section will make the proof in Theorem 1 rigorous in the context of graphs.

▶ **Theorem 11** (Path Connectivity of Graphs Continuously Mapped to $\mathbb{R}^d$). *Let G be a graph. Then, the metric space $(\mathcal{G}_\mathcal{C}(G), d_{FG})$ is path-connected. Moreover, the connected components of the extended metric space $(\mathcal{G}_\mathcal{C}, d_{FG})$ are in one-to-one correspondence with the homeomorphism classes of graphs, making fully rigorous the proof of Theorem 1.*

Proof. Let $\mathbf{G}_0 = (G, \phi_0), \mathbf{G}_1 = (G, \phi_1) \in \mathcal{G}_\mathcal{C}$, for an abstract graph G. Demonstrating the path-connectivity of $\mathcal{G}_\mathcal{C}$ amounts to finding a continuous map $\Gamma\colon I \to \mathcal{G}_\mathcal{C}$ in the extended metric space $(\mathcal{G}_\mathcal{C}, d_{FG})$ such that $\Gamma(0) = \mathbf{G}_0$ and $\Gamma(1) = \mathbf{G}_1$. To define this map Γ, we use linear interpolation:

$$\Gamma(t) := (1-t)\mathbf{G}_0 + t\mathbf{G}_1,$$

where $(1-t)\mathbf{G}_0 + t\mathbf{G}_1$ is a linear combination of $\mathbf{G}_0$ and $\mathbf{G}_1$ (using $c_0 = 1-t$ and $c_1 = t$ in Definition 10). From Definition 10, at any $t \in I$, $\Gamma(t)$ is well defined in the space of continuous mappings since any such linear combination of graphs represents a continuous from the underlying abstract graph to $\mathcal{G}_\mathcal{C}$. Final verification that the constructed Γ is itself continuous is left to Lemma 12.

◀

▶ **Lemma 12.** *The space $(\mathcal{G}_\mathcal{C}(G), d_{FG})$ is path-connected because the map Γ constructed in Theorem 11 is continuous.*

Proof. To see that Γ is continuous, examine an open set $S \subset \mathit{image}(\Gamma)$ given by $S := (S_1 \cap S_2)$ where S_1, S_2 are defined as follows with $\delta_1, \delta_2 < \frac{1}{2}$:

$$S_1 := \{\mathbf{G} \in \mathcal{G}_\mathcal{C}(G) \mid d_{FG}(\mathbf{G}, \mathbf{G}_1) < d_{FG}(((1-\delta_1)\mathbf{G}_0 + \delta_1\mathbf{G}_1), \mathbf{G}_1)$$

$$S_2 := \{\mathbf{G} \in \mathcal{G}_\mathcal{C}(G) \mid d_{FG}(\mathbf{G}, \mathbf{G}_0) < d_{FG}(((1-\delta_2)\mathbf{G}_1 + \delta_2\mathbf{G}_0), \mathbf{G}_0)\}$$

Indeed, S by construction is open in $(\mathcal{G}_\mathcal{C}(G), d_{FG})$. Additionally, S comprises any arbitrary connected open subset of $\mathit{image}(\Gamma)$. By design, $\Gamma^{-1}(S) = (\delta_2, \delta_1) \subset I$, which is open. So, S or any union or finite intersection of open sets S', S'', ... constructed in the same way as S comprises any arbitrary open set in $\mathit{image}(\Gamma)$. Further, since Γ^{-1} acting on any open set is open by design, Γ is continuous, and $(\mathcal{G}_\mathcal{C}(G), d_{FG})$ is path-connected. ◀

C Spaces of Immersions in $\mathbb{R}^d$

Recall that if a path is only locally an embedding, it is called an immersion. More formally, a path $\gamma\colon [0,1] \to \mathbb{R}^d$ *is called an* immersed path *if for any* $t \in (0,1)$ *there exists* $\delta > 0$ *such that* $\gamma|_{(t-\delta,t+\delta)}$ *is injective; see Figure 1. To show path-connectivity of spaces of immersions, the proofs for Theorem 1 and Theorem 11 for continuously mapped paths and graphs* almost *suffice. However, the intermediate paths in* $\Pi_\mathcal{C}$ *and graphs in* $\mathcal{G}_\mathcal{C}$ *might not be immersions. The added steps in the proof sketch for Theorem 2 are sufficient to demonstrate the path connectivity of the topological space* $(\Pi_\mathcal{I}, d_{FP})$*, and here we extend this technique to demonstrate path-connectivity for the space* $(\mathcal{G}_\mathcal{I}(G), d_{FG})$*.*

▶ **Theorem 13** (Path-Connectivity of the Space of Graphs Immersed in $\mathbb{R}^d$)**.** *Let* G *be a graph. Then, the topological space* $(\mathcal{G}_\mathcal{I}(G), d_{FG})$ *is path-connected. Moreover, the connected components of the extended metric space* $(\mathcal{G}_\mathcal{I}, d_{FG})$ *are in one-to-one correspondence with the homeomorphism classes of graphs.*

Proof. Let G an abstract graph and let $\mathbf{G}_0 = (G, \phi_0), \mathbf{G}_1 = (G, \phi_1) \in \mathcal{G}_\mathcal{I}$. As is rigorously described for graphs in Theorem 11, construct a continuous path $\Gamma : [0,1] \to \mathcal{G}_\mathcal{I}$ such that $\Gamma(0) = \mathbf{G}_0$ and $\Gamma(1) = \mathbf{G}_1$ by interpolating along the pointwise matchings defining $d_{FG}(\mathbf{G}_0, \mathbf{G}_1)$. (Which is to say, interpolate along the linear combinations of $\mathbf{G}_0$ and $\mathbf{G}_1$ as defined in Definition 10.) However, as in Theorem 2, there may exist $t \in [0,1]$ where a self-crossing event could occur. Again, we must ensure that such an event does not result in any edge degeneracies, which would imply $\Gamma(t) = \mathbf{G}_t \notin \mathcal{G}_\mathcal{I}(G)$. At this juncture, there must exist $t - \epsilon$ for sufficiently small $\epsilon > 0$ where $\mathbf{G}_{t-\epsilon}$ is near enough to the crossing event at $\mathbf{G}_t$ not to create any new crossings when conducting the inflation and self-crossing steps described in Theorem 2 and depicted in Figure 2c and Figure 2d.

Denote the images of the edge $e \in E \subset G$ and its two corresponding vertices to be $\mathbf{e}_0 = (e, \phi_0) \subseteq \mathbf{G}_0, \mathbf{e}_1 = (e, \phi_1) \subseteq \mathbf{G}_1$, and $\mathbf{e}_{t-\epsilon} = (e, \phi_{t-\epsilon}) \subseteq \mathbf{G}_{t-\epsilon} = \Gamma(t-\epsilon)$. Suppose the crossing event were to occur due to the interpolation along $\mathbf{e}_t = (e, \phi_t)$. As in Theorem 2, we denote the exact point corresponding to the crossing event in $\mathbf{e}_t$ as $\phi_t(t^*)$ for $t^* \in [0,1]$, where $||\phi_{t-\epsilon}(t^*) - \phi_t(t^*)||_{inf} = \delta$ for $\delta > 0$. Then, we suspend all interpolation at time $t - \epsilon$, and inflate a small region of the image of $\phi_{t-\epsilon}$ to share equivalent pointwise leash-length distances to $\phi_{t-\epsilon}(t^*)$, where this neighborhood is defined by $\phi_{t-\epsilon}|_{(t^*-\delta^*,t^*+\delta^*)}$ for $\delta^* > 0$ and $(t^*-\delta^*, t^*+\delta^*) \subset [0,1]$. Here, we define δ^* to be small enough again not to cause any additional crossing events. That is, if $x \in \phi_{t-\epsilon}|_{(t^*-\delta^*,t^*+\delta^*)}$, then $d_{FG}(x, \mathbf{e}_1) = d_{FG}(\phi_{t-\epsilon}(t^*), \mathbf{e}_1)$. This is done, analogously to the procedure shown Figure 2c, in order to avoid strictly increasing the Fréchet distance when constructing a path in the space $(\mathcal{G}_\mathcal{E}(G), d_{FG})$.

Finally, directly perturb $\phi_{t-\epsilon}|(t^*)$ by 2δ so that the edge crossing event occurs, as in Figure 2d, and $\phi_t(t^*)$ lies distance δ on the other side of the original edge crossing point if interpolation would've been followed. Following this crossing event, continue linear interpolation as prescribed in Definition 10, handling subsequent crossing events in the same manner. After all crossings have occured, linear interpolation will attain $\mathbf{G}_1$. Hence, the space $(\mathcal{G}_\mathcal{I}(G), d_{FG})$ is path-connected. ◀

D Spaces of Embeddings in $\mathbb{R}^d$

The ideas presented in Theorem 3 and Theorem 4 are sufficient to demonstrate the path-connectedness property in each corresponding topological space. In this section, we make *rigorous the proof sketch in Theorem 3 and further elaborate upon the proof sketch in Theorem 4, in order to formalize each proof.*

▶ **Theorem 14** (Path-Connectivity of the Space of Paths Embedded in $\mathbb{R}^d$). *The space of curves embedded in $\mathbb{R}^d$ under the Fréchet distance, $(\Pi_\mathcal{C}, d_{FP})$, is path-connected.*

Proof. Without loss of generality, we need to construct a continuous $\Gamma : I \to \Pi_\mathcal{E}$ in the extended metric space $(\Pi_\mathcal{E}, d_{FG})$ such that $\Gamma(0) = \phi_0$ and $\Gamma(1) = \phi_1$. To begin, define $\Gamma_0 : I \to \Pi_\mathcal{E}$, and $\Gamma_1 : I \to \Pi_\mathcal{E}$, by restricting the domains of ϕ_0, and ϕ_1, thereby condensing each curve toward its center:

$$\Gamma^0_s(t) := \phi_0|_{[s/2,1-s/2]}(t)$$

$$\Gamma^1_s(t) := \phi_1|_{[s/2,1-s/2]}(t)$$

Then, as $t \to 1$, the images of ϕ_0 and ϕ_1 encompass an increasingly smaller, and therefore straighter curve in the embedding space. As a consequence of Taylor's theorem, both images must attain some juncture at time t^*_0 and t^*_1 where $\phi_0|_{(t^*_0/2,1-t^*_0/2)}$ and $\phi_1|_{(t^*_1/2,1-t^*_1/2)}$ can be continuously straightened in $\Pi_\mathcal{E}$ toward the line tangent to the center of each curve. From there, a standard interpolation between straight segments may be used to transform the remaining image of ϕ_0 to ϕ_1. Consequently, we obtain the desired Γ by the composition of the condensing maps Γ^0_s and $\Gamma^1_s(t)$, and the straightening and linear interpolation steps once each condensing map has attained the restriction $\phi_0|_{(t^*_0/2,1-t^*_0/2)}$ and $\phi_1|_{(t^*_1/2,1-t^*_1/2)}$.

Note that the requirement in Section 2 that ϕ_0 and ϕ_1 are rectifiable is crucial for the above construction. Were this not the case, there would be no guarantee that one could condense the images of ϕ_0 and ϕ_1 to become "straight enough" in order to continuously achieve a straight segment in the space $\Pi_\mathcal{E}$.

◀

▶ **Theorem 15** (Path-Connectivity of the Space of Graphs Embedded in Low Dimensions). *In general, the topological space $(\mathcal{G}_\mathcal{E}(G), d_{FG})$ is not path-connected for any arbitrary abstract graph G, if G is embedded in $\mathbb{R}^d$ with $d \leq 3$.*

Proof. If embeddings in $\mathbb{R}^d$ are restricted to $d \leq 3$, then as a consequence of knot theory, $(\mathcal{G}_\mathcal{E}(G), d_{FG})$ is not path-connected for any abstract graph G.

If $d = 2$, let G denote an abstract graph consisting of only a cycle comprising two vertices, and a single dangling edge. Let $\mathbf{G}_0 = (G, \phi_0) \in \mathcal{G}_\mathcal{E}$ comprise a closed curve with an interior edge, and let $\mathbf{G}_1 = (G, \phi_1) \in \mathcal{G}_\mathcal{E}$ comprise a closed curve with an exterior edge in $\mathbb{R}^d$. By the Jordan curve theorem, there does not exist a continuous path in $\mathbb{R}^d$ from $\mathbf{G}_0$ to $\mathbf{G}_1$ that does not create a degeneracy. Then, constructing a path from $\mathbf{G}_0$ to $\mathbf{G}_1$ must reach some juncture where an immersed graph in $\mathbb{R}^d$, denoted $\mathbf{G}_* = (G, \phi^*_0)$, is not homeomorphic to G. Therefore, $\mathbf{G}_*$ violates the definition of a graph embedding, and the space $(\mathcal{G}_\mathcal{E}, d_{GF})$ is not path-connected among homeomorphism classes of graphs in dimension 2.

If $d = 3$, let G consist of a single cycle, and $\mathbf{G}_0 = (G, \phi_0) \in \mathcal{G}_\mathcal{E} = \mathbb{S}^1$ and $\mathbf{G}_1 = (G, \phi_1) \in \mathcal{G}_\mathcal{E}$ comprise a trefoil knot. Then, again due to the Jordan curve theorem and elementary knot theory, there exists no continuous path from $\mathbf{G}_0$ to $\mathbf{G}_1$ in the space $(\mathcal{G}_\mathcal{E}(G), d_{FG})$.

◀

▶ **Theorem 16** (Path-Connectivity of the Space of Graphs Embedded in Higher Dimensions). *In general, the topological space $(\mathcal{G}_\mathcal{E}(G), d_{FG})$ is path-connected for any arbitrary abstract graph G, if G is embedded in $\mathbb{R}^d$ with $d \geq 4$. Moreover, the connected components of the* *extended metric space* $(\mathcal{G}_{\mathcal{I}}, d_{FG})$ *are in one-to-one correspondence with the homeomorphism classes of graphs.*

Proof. Let G and abstract graph, and $\mathbf{G}_0 = (G, \phi_0), \mathbf{G}_1 = (G, \phi_1) \in \mathcal{G}_{\mathcal{E}}$. In dimension 4 or higher, it is well known that any tame knot can be unwound by a sequence of Reidemeister moves into the unknot. Then, one may interpolate along the pointwise matchings (leashes) defining $d_{FG}(\mathbf{G}_0, \mathbf{G}_1)$ until a crossing event must occur. At this juncture, there must exist a Reidemeister move allowing the crossing event to occur. Hence, any sequence of knots and dangling edges comprising the image of ϕ_0 can be unwound to a sequence of unknots and straight edges. The same holds for the image of ϕ_1. Consequently there exists a continuous path from $\mathbf{G}_0$ to $\mathbf{G}_1$ in the topological space $\mathcal{G}_{\mathcal{E}}(G, d_{FG})$. Note that we require that ϕ_0, ϕ_1 are rectifiable in Section 2, which validates the above argument. Without this requirement, $\mathbf{G}_0$ and $\mathbf{G}_1$ could comprise wild knots, and constructing such a path could consist of infinitely many Reidemeister moves. ◀