Geometry In Coordinates, 3.5

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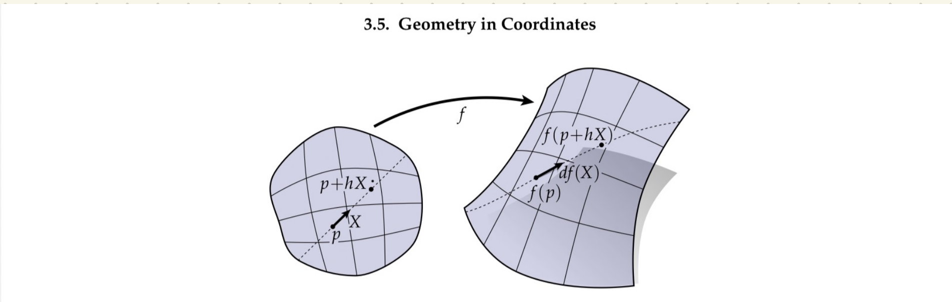
3.5 Geometry In Coordinates

So far: Abstract def'ns of objects

"df of an immersion f tells us how to stretch tangent vectors from domain to image"



Can be more precise and give $df(X)$ w/ limits:

$$df_p(X) = \lim_{h \to 0} \frac{f(p+hX) - f(p)}{h}$$

More precise still, talk about matrices.

Suppose $\underline{f: M \to \mathbb{R}^3}$ an immersion.

Can take df as the Jacobian

$$J = \begin{bmatrix} \partial f^1/\partial x^1 & \partial f^1/\partial x^2 \\ \partial f^1/\partial x^1 & \partial f^2/\partial x^2 \\ \partial f^3/\partial x^1 & \partial f^3/\partial x^2 \end{bmatrix}$$

where

$$f(x^1, x^2) = (f_1(x^1, x^2), f_2(x^1, x^2), f_3(x^1, x^2))$$

for scalars $f_1, f_2, f_3 : M \to \mathbb{R}$,

So $df(X)$ is J applied to a vector $\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$.

Drawbacks to this approach:

What's this matrix represent?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

adjacency matrix

Pauli matrix

represents spin on

x-axis

Dihedral group

D_4?

Easy to forget where matrices

come from

The real philosophical point here is that *matrices are not objects: they are merely* ***representations*** *of objects!* Or to paraphrase Plato: matrices are merely shadows on the wall of the cave, which give us nothing more than a murky impression of the real objects we wish to illuminate.

Eg) Linear operators vs bilinear forms:

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $u \to f(u)$

vector space to vector space

$A \to PAP^{-1}$

$g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \to g(u,v)$

pair of vectors to a scalar

$B \to P^T B P^{-1}$

These, as matrices, behave differently w.r.t. change of basis!

Give it a minute, we'll be back...

Standard Matrices in the Geometry of Surfaces

Still, be aware of matrix representations in geometry.

Eg) - The differential can be encoded as J
- Induced metric g:

Just a function of differential!

Recall) $g(u,v) = df(u) \cdot df(v)$

i.e. I_M

$$u^T I_M v = (Ju)^T (Jv)$$

so $I_M = J^T J$

Remark: Turns out, in old books,

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Another Eg: Shape operator $S : TM \to TM$

(tangent bundle)

s.t. $dN(X) = df(SX)$

(Weingarten Map)

and second fundamental form:

$\mathrm{II}(u, v) = g(S_u, v)$

Taking $S, \mathrm{II} \in \mathbb{R}^{2\times 2}$,

$$u \,\mathrm{II}\, v = u^T I_M S v$$

$$\Rightarrow \mathrm{II} = I_M S$$

Rmk: $\mathrm{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$, $e = N \cdot f_{xx}$, $f = N \cdot f_{xy}$, $g = N \cdot f_{yy}$

N is unit surface normal

f_{xy} is second partial along x, y.

How does $I_\mu S$ transform w/ a change of basis?

I_μ is a bilinear form, S is a linear map.

↳ can't tell by staring at matrices, but what $\underset{S}{\text{are}}$ these things??

IS corresponds to second fundamental form, so it should ~~transform~~ transform like any old bilinear form

$$I_\mu S \to P^t I_\mu S P^{-1}$$

w.r.t. change-o-basis P.

Final Extra credit Verifications:

Normal curvature is classically:

$$\kappa_n(u) = \frac{\mathrm{II}(u,u)}{\mathrm{I}_M(u,u)}$$

change in normal direction

$$= \frac{u^T \mathrm{II}\, u}{u^T \mathrm{I}_M u} = \frac{u^T \mathrm{I} S\, u}{u^T \mathrm{I}_M u} = \frac{(Ju)^T (JSu)}{(Ju)^T (Ju)} = \frac{df(u)\, dN(u)}{|df(u)|^2}$$

Same one we saw w/

curves embedded in surfaces!