Topological Properties of Fréchet Spaces

With Dr. Fasy, Dr. Chambers, Dr. Wenk, Dr. Majhi

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Overview:

Fréchet Distance

- For paths
- For graphs

Different Spaces

- Continuous mappings
- immersions
- embeddings

Path connectivity

- proof sketch for each space
- Discuss path-connectivity of open balls

Recall: Fréchet Distance for paths

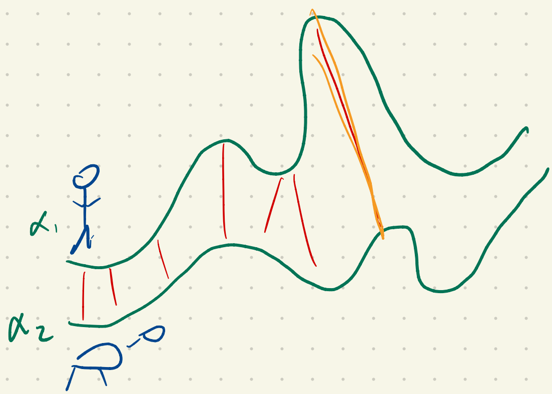
Any continuous map $\alpha : I \to \mathbb{R}^d$ is a path in $\mathbb{R}^d$

The Fréchet distance between two paths α_1, α_2 living in $\mathbb{R}^d$ is

$$d_{FP}(\alpha_1, \alpha_2) = \min_{r: I \to I} \max_{t \in I} |\alpha_1(t) - \alpha_2(r(t))|$$

(where r is all reparametrizations of the interval I)

Eg:)



Frechet Distance for spaces of Graphs

Could (carefully*) do an analogous thing for graphs.

Let G a graph, and $\varphi_1, \varphi_2 : G \to \mathbb{R}^d$ continuous, rectifiable maps.

Given homeo. $h : G \to G$, call induced L_∞ distance between $\varphi_1, \varphi_2 \circ h$

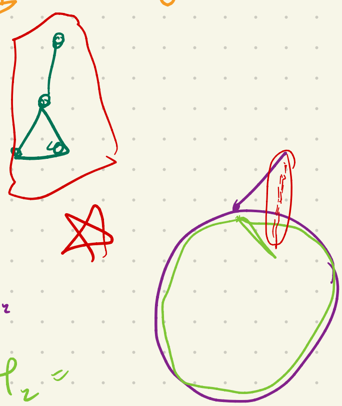
$$\| \varphi_1 - \varphi_2 \circ h \|_\infty = \max_{x \in G} | \varphi_1(x) - \varphi_2(h(x)) |$$

and then the graph Frechet distance is

$$d_{FG}\big((G, \varphi_1), (G, \varphi_2)\big) = \min_h \| \varphi_1 - \varphi_2 \circ h \|_\infty$$

* one of multiple ways to do this!

Eg: $G =$



$\varphi_1 =$

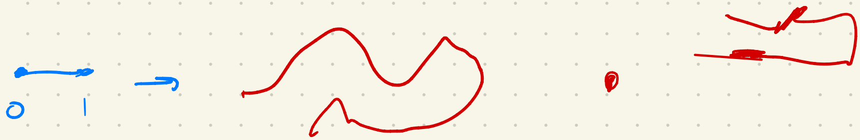
$\varphi_2 =$

So, what does "living in" $\mathbb{R}^d$ mean for a path / graph?

Say for a path $\alpha : I \to \mathbb{R}^d$.

3 different definitions:

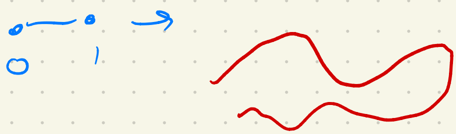
1.) α just needs to be continuous Π_C



2.) α needs not only continuity, but injectivity

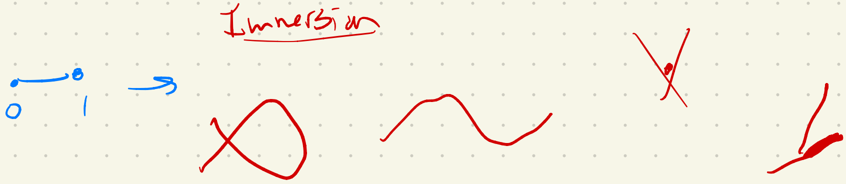
Embedding

Π_E



3.) α needs to be injective, but only locally.

Immersion

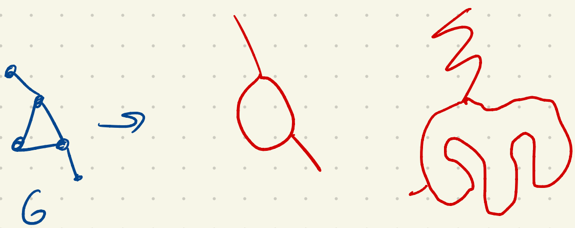


Ditto for graphs.

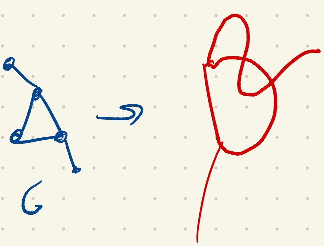
Could require, for a map $\varphi : G \to \mathbb{R}^d$

1.) Just that φ is cont's $\quad G_C$

2.) That φ is 1-1 $\quad G_E \quad$ Embedding



3.) That φ is only locally 1-1 $\quad$ Immersion



G_I

Are these spaces, topologized by their respective Fréchet distances, path-connected?

i.e. For any $x_0, x_1 \in X$

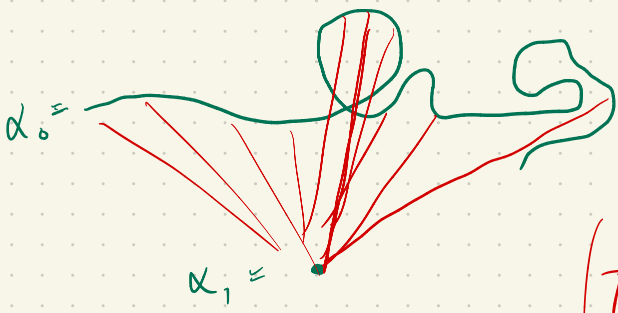
can we construct a continuous $\Gamma : [0,1] \to X$

such that $\Gamma(0) = x_0$ and $\Gamma(1) = x_1$??

Π_c : (Space of continuously mapped paths in $\mathbb{R}^d$)

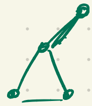
Yep. Just interpolate btwn $\alpha_0, \alpha_1 \in \Pi_c$.

In an image:



(Π_c, d_F) ✓

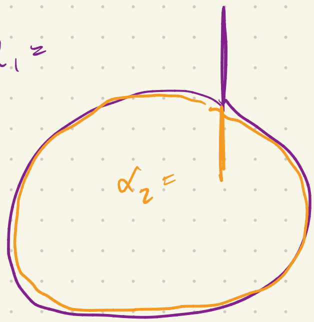
G_c: Ditto for graphs; just interpolate along leashes.

Eg.) Let $G =$ 

at time $t=0$:

$\alpha_1 =$

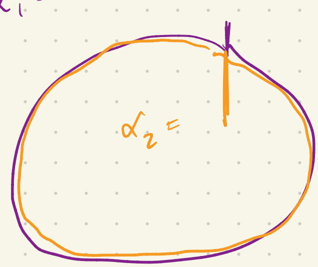
$\alpha_2 =$



$t = \frac{1}{2}$

$\alpha_1 =$

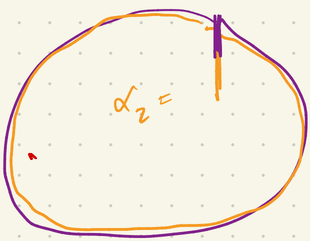
$\alpha_2 =$



$t = 3/4$

$\alpha_1 =$

$\alpha_2 =$



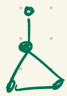
$t = 1$

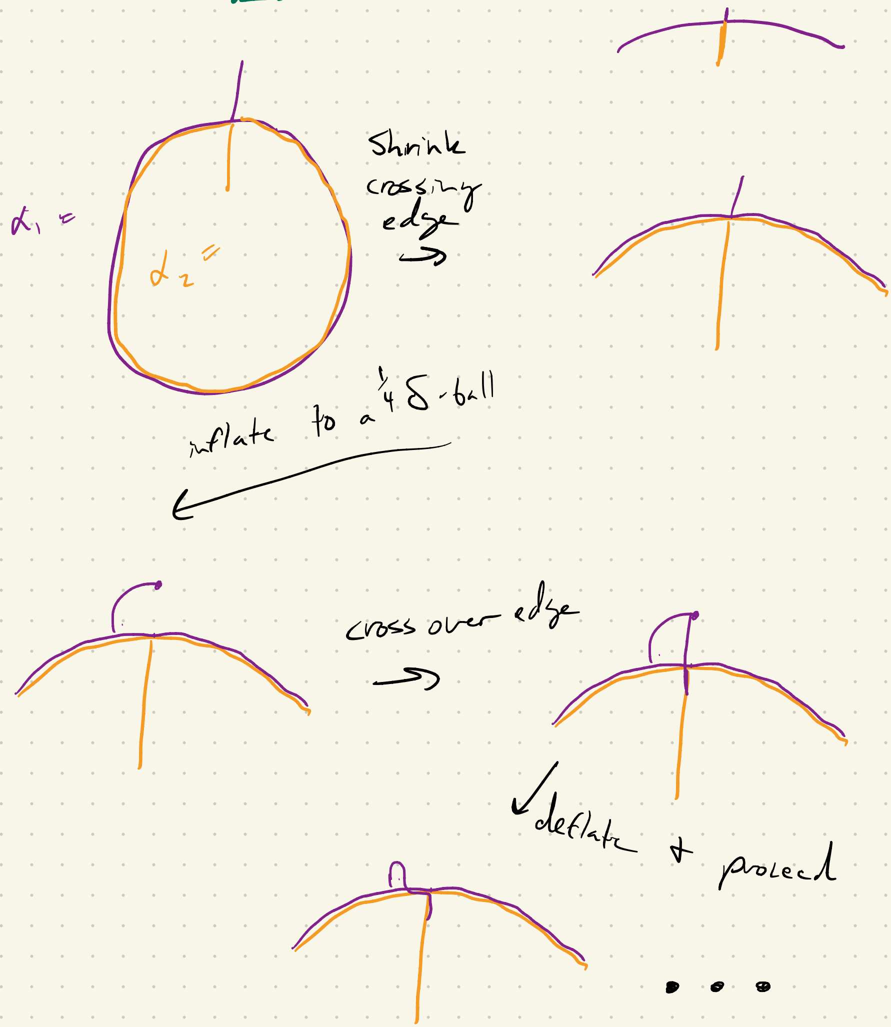
$=$

What if we restrict to inversions?

Need to be more careful about crossing over ourselves.

Sketch of proof in a picture:

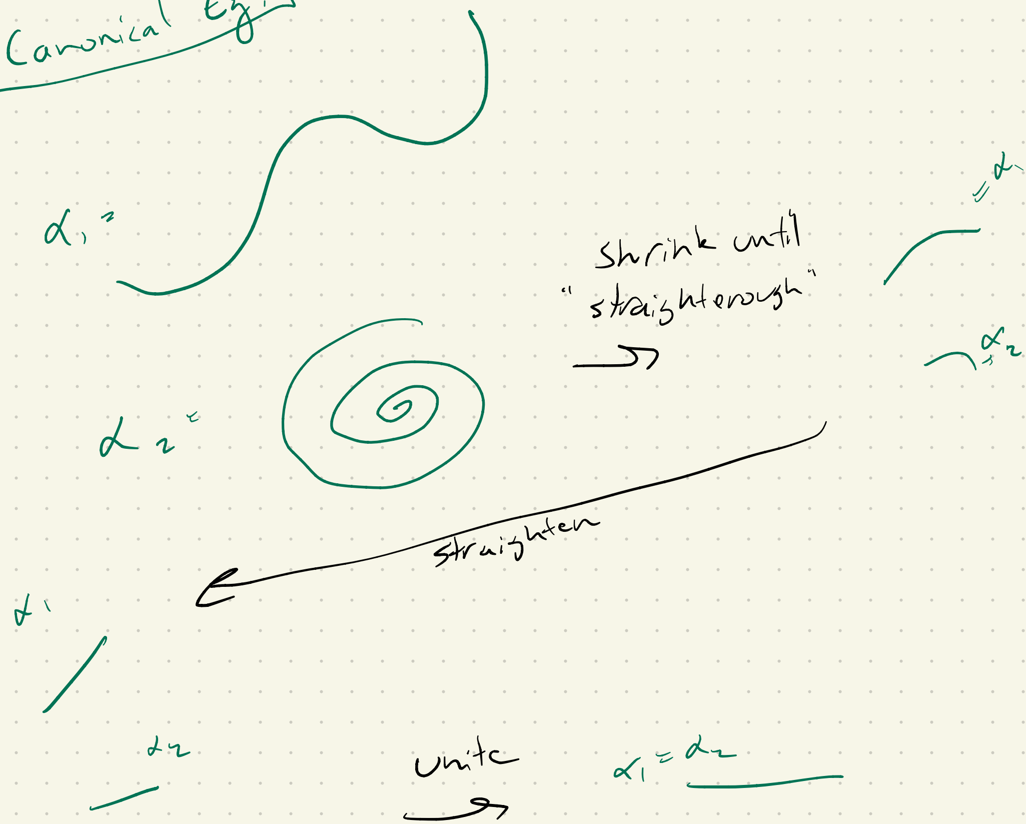
E.g.) Let $G =$ 



What if we restrict to embeddings?

How about Π_ε, the space of paths <u>embedded</u> in $\mathbb{R}^d$?

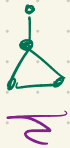
Canonical Eg:

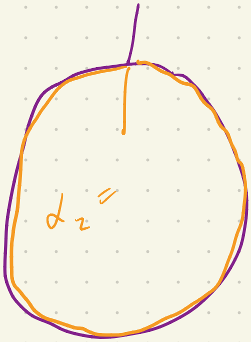


Rectifiability Needed here

What about embedded graphs G_ε?

Eg)

Let $G =$  as before, and

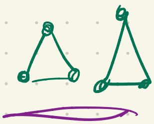
$\alpha_1 =$ 

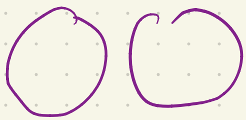
If we're restricted to $\mathbb{R}^2$, $\nexists$ a path in G_ε from α_1 to α_2 by Jordan curve theorem.

Pretty soon, this turns into knot theory.

What if we restrict G_ε to dimension 3?

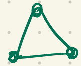
space of all graphs embedded in $\mathbb{R}^3$ under Fréchet graph distance.

Eg:) Suppose $G =$  and

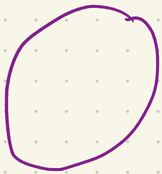
$\alpha_1 =$  while $\alpha_2 =$ 

Then α_1 + α_2 ain't path connected by knot theory.

What if we're restricted to $\beta_0 = 1$?

Eg) Now suppose $G =$  and

φ

$\alpha_1 =$  while $\alpha_2 =$ 

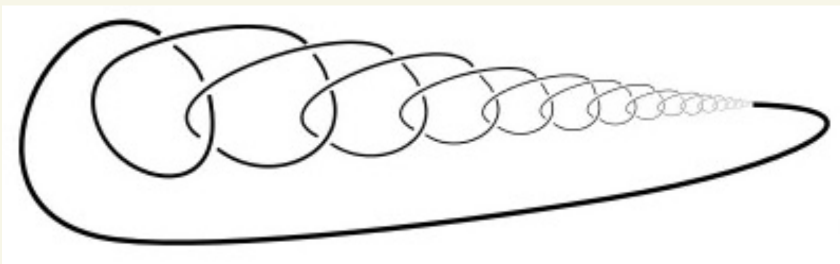
well then still, in $\mathbb{R}^3$, the space G_ε isn't path-connected, by knot theory.

So how about for $\mathbb{R}^d$, $d \geq 4$?

In general, all such TAME 1-dimensional knots are unknotted in $\mathbb{R}^4$, which fixes this dilemma.

However, one must be careful.

A WILD example!

$\alpha_1 =$ 

Can't be unknotted in $\mathbb{R}^4$ (would take infinite moves)!

So again rectifiability is required.

G_ε

The next question....

Are open balls in any of these spaces path-connected?

This is to say, if $d_F(\alpha_1, \alpha_2) < \delta$,

does there exist a path maintaining consistently this distance?

Using earlier ideas, we can't quite pull this off (yet) with embeddings.

E.g.: α_1:

$= \alpha_2$

The rest to be continued!