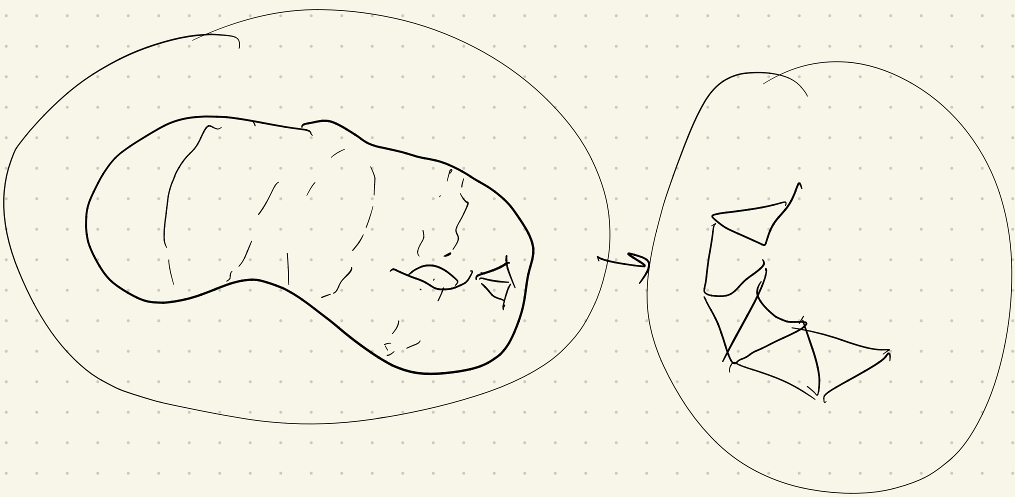
Discrete Exterior Calculus

4.8

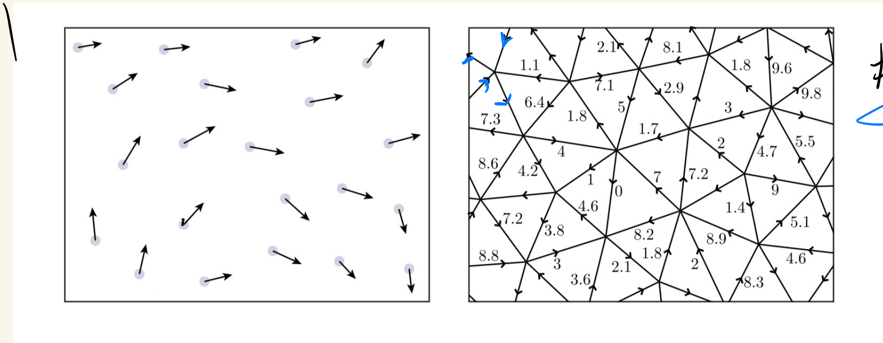
8/12/2022

Motivation

- So far, we've seen exterior calculus in just a smooth setting.
- Want for a computer to do so in discrete (finite) setting.
- Discrete Exterior Calculus (DEC) is just an extension of continuous calculus to a mesh.



4.8.1 Discrete Differential Forms

#1  #2

How to encode a 1-form on a surface?

- Integrate 1-form over each edge of a mesh
- Store resulting numbers as edge weights

That is, for α a 1-form and e an edge,

$$\hat{\alpha}_e := \int_e \alpha$$

is a scalar edge weight.

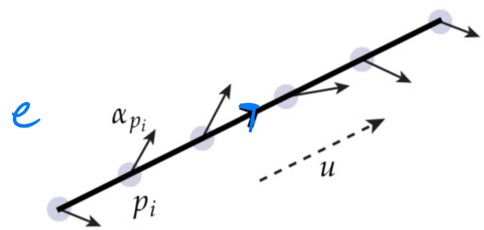
Tells us how strongly α flows along e on average

$\hat{\alpha}_e = \int_e \alpha$ is computed by:

1) finding tangent vector to all pts in e.
2) Stick tangent vector in 1-form → record for e
3.) Sum up result → edge weights

$$\int_e \alpha \approx |e| \left(\frac{1}{N} \sum_{i=1}^{N} \alpha_{p_i}(u) \right),$$ → unit tangent vector to e

where $|e|$ denotes the length of the edge, $\{p_i\}$ is a sequence of points along the edge, and $u := e/|e|$ is a unit vector tangent to the edge:



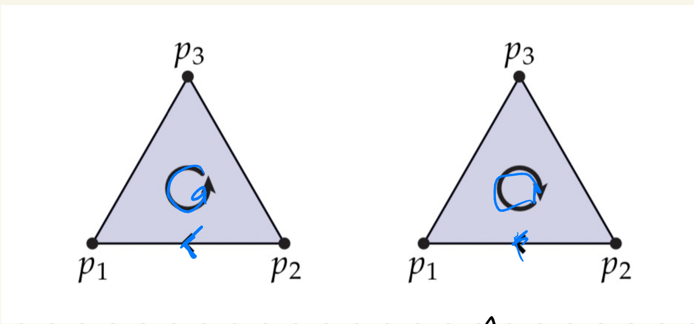
Rmk 1: Doesn't tell us about the flow orthogonal to the edge! → assuming that adjacent edges will pick up on orthogonal flows

Rmk 2: Can be done analogously for k-forms by integrating over each k-cell

Orientation

$$\int_a^b \frac{\partial \phi}{\partial x} dx = \phi(b) - \phi(a) = -(\phi(a) - \phi(b)) = -\int_b^a \frac{\partial \phi}{\partial x} dx.$$

- It's not enough to integrate only, we must include orientation on edges.
- By extension, we must do the same for k-forms by orienting k-simplices



Orientation "agrees" between faces if a face has same ordering as its simplex.

Eg. $\{p_2, p_1\}$ and $\{p_1, p_3, p_2\}$

How to integrate a k-form?

Recall: k-form "eats" k-vectors & spits out a scalar.

Canonical Solution:

- Take ordered collection of k-vectors
- Orthogonalize them (Using Graham-Schmidt)
- Numerically approximate by the following:

$$\int_\sigma \alpha \approx \frac{|\sigma|}{N} \sum_{i=1}^{N} \alpha_{p_i}(u_1, \ldots, u_k)$$

where u_i are orthogonal vectors at points p_i

Rmk: How to integrate a 0-form?

- Must integrate over all vertices
- Integral of 0-simplex is just value of the function at that point
- Always positive ☆ all permutations even!

The Discrete Exterior Derivative

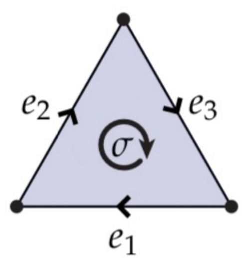
Recall Stokes Theorem: $\displaystyle\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$

for any k-form & $k+1$ dimensional domain Ω.

↳ which is to say, we can integrate the derivative of a differential form if we know its integral along the boundary.

Eg) If $\hat{\alpha}$ is stored on the edges of a Δ:

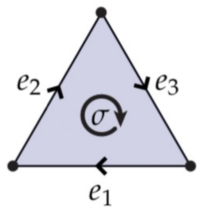
$$\int_{\sigma} d\alpha = \int_{\partial\sigma} \alpha = \sum_{i=1}^{3} \int_{e_i} \alpha = \sum_{i=1}^{3} \hat{\alpha}_i.$$



we can exactly evaluate the integral by adding just 3 numbers!

2 form "$\hat{d}\alpha$" integrated over our triangle

$$\int_\sigma d\alpha = \int_{\partial\sigma} \alpha = \sum_{i=1}^{3} \int_{e_i} \alpha = \sum_{i=1}^{3} \hat{\alpha}_i.$$



call $\hat{d}$ the Discrete Exterior Derivative

What's this do?

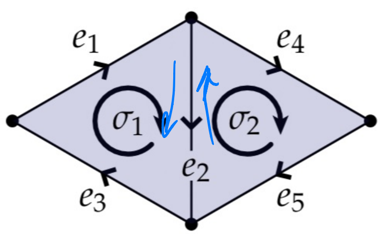
↳ Gives us a derivative in a dimension higher!

Rmk:

Not so simple as summing up edge weights, though!

$$(\hat{d}\hat{\alpha})_1 = \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$$

$$(\hat{d}\hat{\alpha})_2 = \hat{\alpha}_4 + \hat{\alpha}_5 - \hat{\alpha}_2.$$



Issue: discrete k-form captures the behavior of a continuous k-form along k directions, but not along remaining n-k directions.

→ Want to get to a notion of Hodge duality

k form → n-k form

→ Need to construct a dual mesh

↓

Identify each k-simplex with an unique n-k simplex.

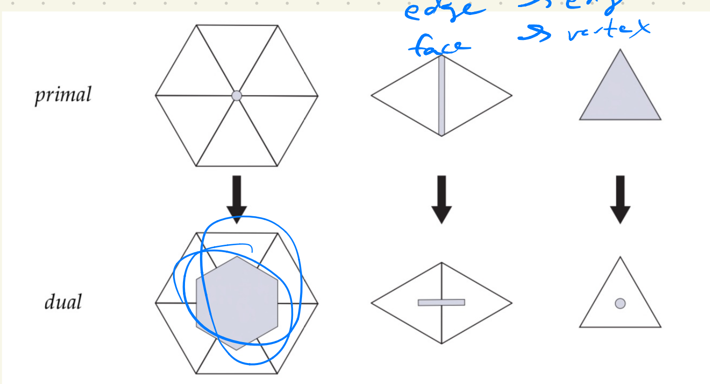
In two dimensions,

vertex → face

edge → edge

face → vertex

Eg.)



Discrete Hodge Star:

In a 2d simplicial mesh:

√ vertices → faces

√ edges → edges

√ faces → vertices

↳ may require that pairings inhabit orthogonal linear subspaces

↓

naturally leads to

Discrete Hodge Dual of a k-form on primal mesh is a n-k-form on dual mesh

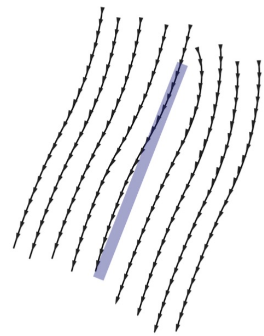
Given a discrete form $\hat{\alpha}$, write its Hodge Dual $\hat{\star}\hat{\alpha}$

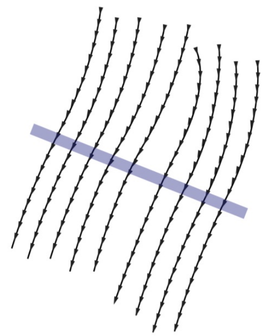
primal k-mesh ⟶ dual n-k mesh

↓ ↓

weights on k-mesh ⟶ n-k weights on dual

Primal & Dual k-forms live in different places, and have different physical interpretations





primal 1-form (circulation) dual 1-form (flux)

Naturally leads to diagonal Hodge Star:

consider primal k-form α. If $\hat{\alpha}_i$ is value of $\hat{\alpha}$ on a simplex σ_i:

$$\hat{\star}\hat{\alpha}_i = \frac{|\sigma_i^\star|}{|\sigma_i|}\hat{\alpha}_i$$

n-k form (dual)

primal k-form

Diagonal Hodge Star

$$\hat{\star}\hat{\alpha}_i = \frac{|\sigma_i^\star|}{|\sigma_i|}\hat{\alpha}_i$$

In words, to get the dual form we just take the scalar on each simplex multiplied by ratio of corresponding # of simplices in dual vs primal meshes.

k-form start

end n-k form

called "diagonal" since the ith element of the dual differential form depends only on ith element of the primal differential form.

That's about all from 4.8!