Generating Discrete Morse Functions in Near Linear Time

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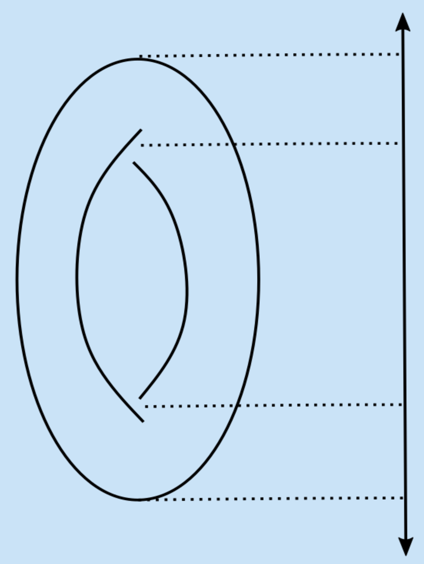
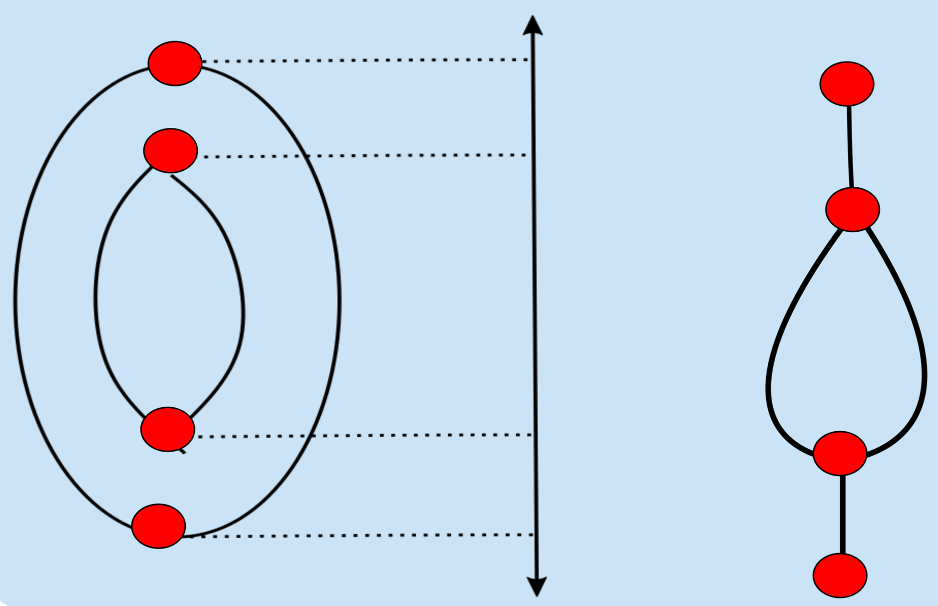
Background

Classical Morse theory:

If we're given a smooth manifold, we may interpret the topology of that manifold using a function to the real numbers.

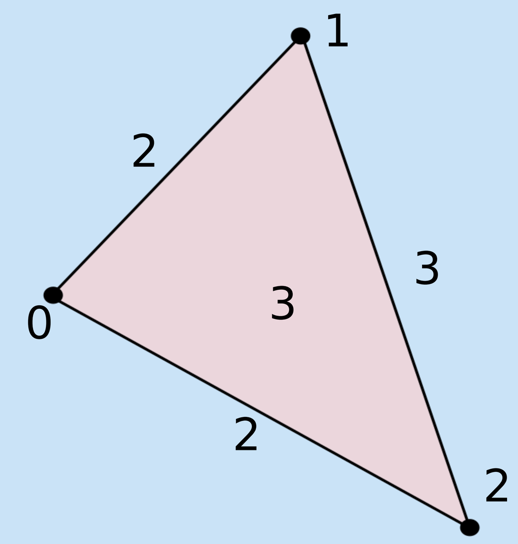


From this function, the critical points tell the story:



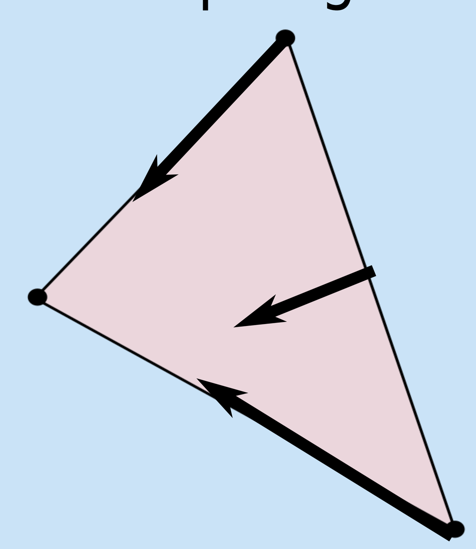
Three (Equivalent) Flavors of Discrete Morse Theory

Algebraic



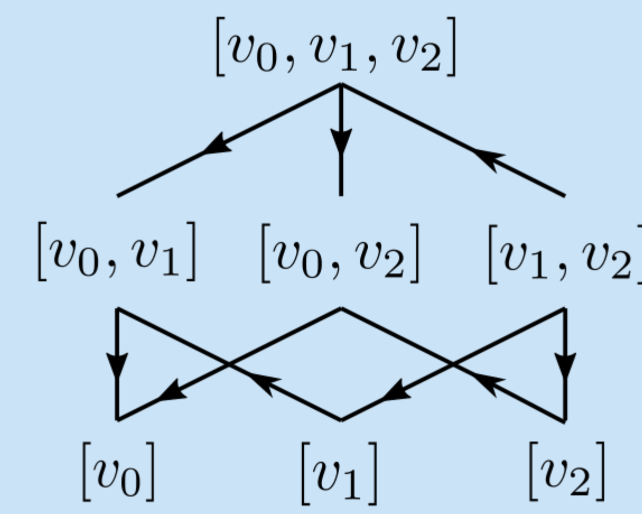
Function values increase as dimension increases, with at most one exception per simplex

Topological



A gradient vector field is defined on the complex, with matchings between faces and cofaces

Combinatorial



Acyclic matchings in the Hasse diagram of the complex dictate the function

Importantly, the number of degree-i critical (unmatched) simplices bounds the ith Betti number,giving valuable insight towards the ith homology group!

Ongoing work:

- Ideally, we would reduce the entire algorithm of King et al. to Θ(dn). To do so we conjecture new steps to cancel extranneous critical cells

- Naive cancellation possible when there are unique gradient paths from a critical j-simplex to a critical j-1 simplex.

- We cancel by simply reversing a gradient path, thereby making both participating critical cells the contents of a matching in the complex.

- Use a modified union-find data structure to avoid quadratic time.

We pose an additional problem:

- What if data is dynamic? That is, what if we do any of three things:

  1.) Permute function values on vertices?
  2.) Add a new simplex at will?
  2.) Delete a simplex at will?

- A new paper is in the works to address these problems. We conjecture the properties of our newest algorithm ExtractRightChild are the key to these additional problems.

Lastly, an implementation for our algorithm is available at: https://github.com/compTAG/morse-alg/tree/master/code

Our paper published in CCCG is available on Arxiv: https://arxiv.org/abs/2103.13882

And a video of our CCCG talk is available here: https://www.youtube.com/watch?v=kHpD-J4EzI8

Introduction

Discrete Morse theory:

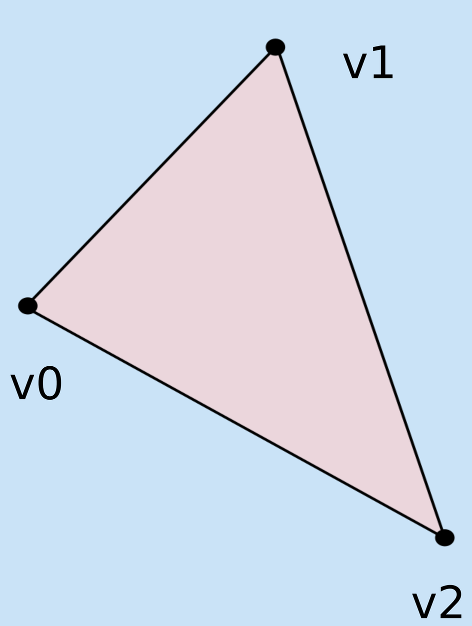
Can we take the ideas from smooth Morse theory and apply them to data?

In 1998 Forman shows that many of the tools of conventional Morse theory may be adapted to the discrete setting.

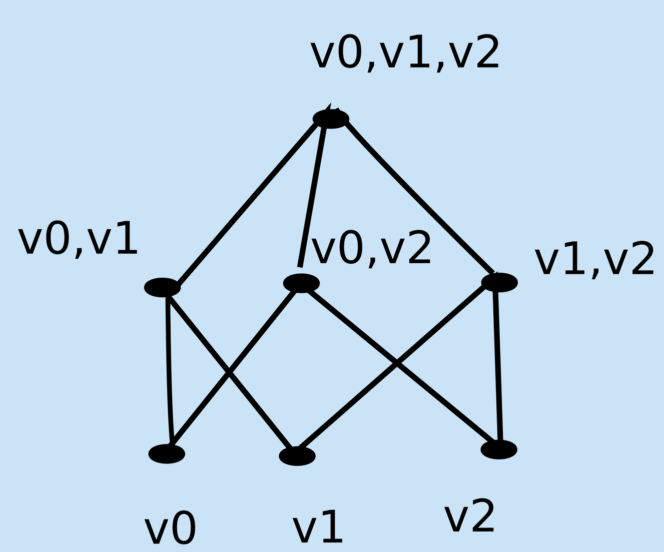
For us, this focuses on adapting Morse theory to a simplicial complex:

Simplicial Complex Definition by Example:

Conventionally:



Combinatorially:



Recent Contributions

- King et al. show that a discrete Morse function with data on vertices can be generated in polynomial time, Θ(n^2log(n)).

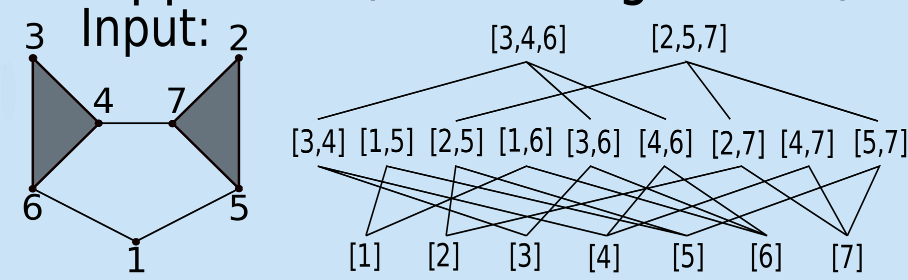
- We reduce this to Θ(dn), where d is the total dimension of the complex and n is the total number of simplices. We use combinatorial properties for our algorithm.

- Generally, findinig the best possible discrete Morse function is NP-Hard

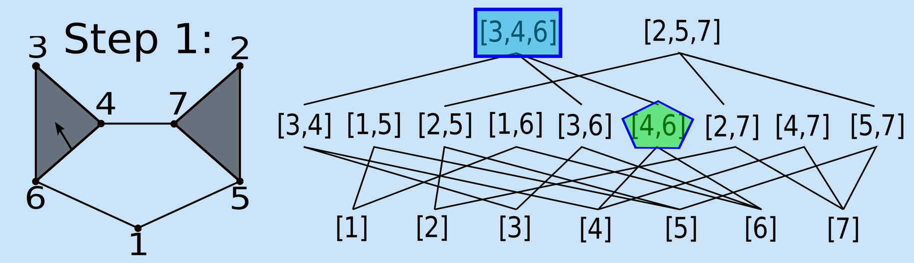
- Improving the runtime to generate Morse functions could prove vital in wide ranging computational topology applications. Historically, TDA has been limited by large, high dimensional data. Morse theory may be the remedy to this problem.

Our Approach (ExtractRightChild):

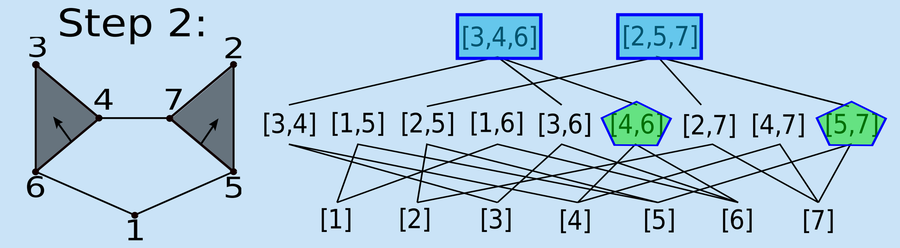
Input:



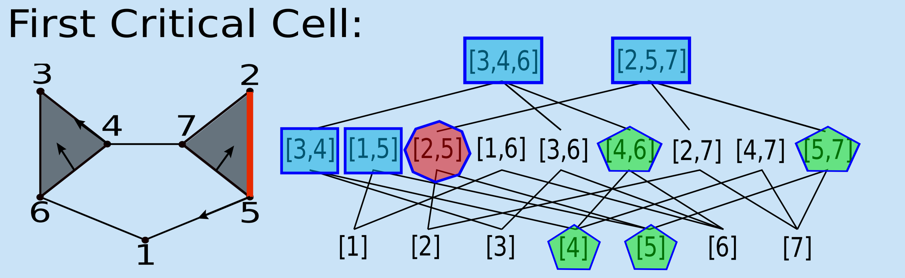
Step 1:



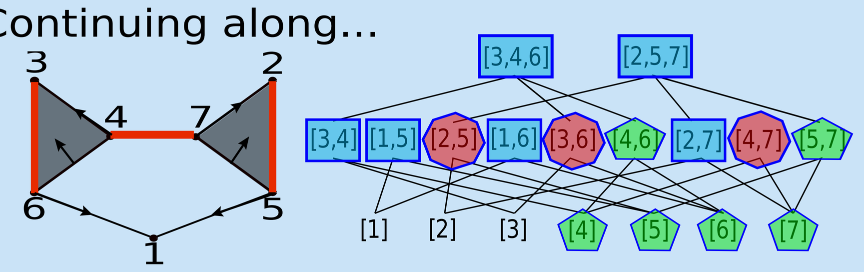
Step 2:



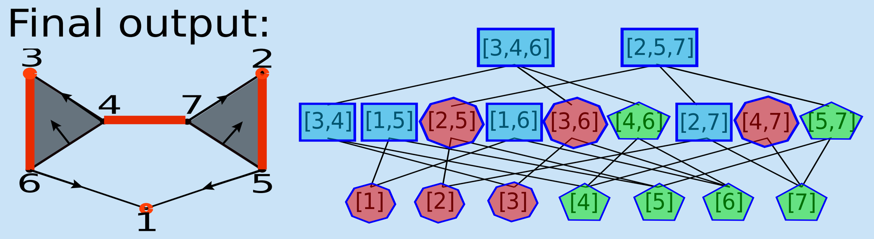
First Critical Cell:



Continuing along...



Final output:



Our algorithm greedily exploits an invariant of ExtractRaw, a major subalgorithm of King et al.

We produce a rudimentary discrete Morse function in this way. However, we may find extranneous critical cells. Cancelling these simplices is a current area of research.

Works Cited:

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