If You Must Choose Among Your Children, Pick the Right One

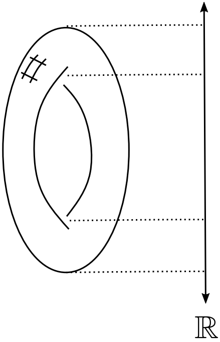
Brittany Terese Fasy Benjamin Holmgren
Bradley McCoy David L. Millman

Montana State University

4-7 August 2020, Saskatoon
CCCG 2020

Smooth Morse Theory

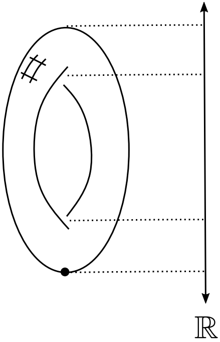
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

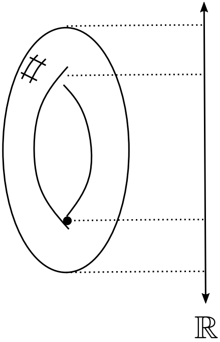
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

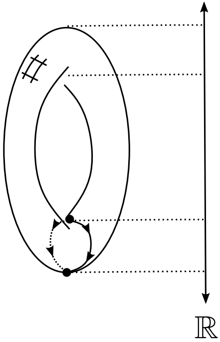
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

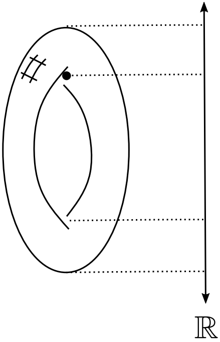
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

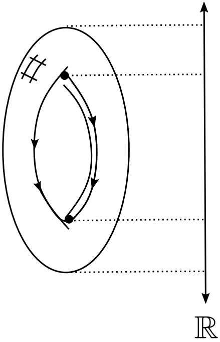
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

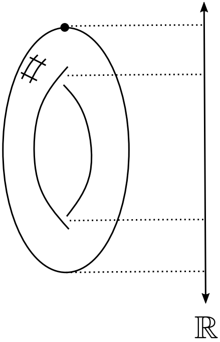
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

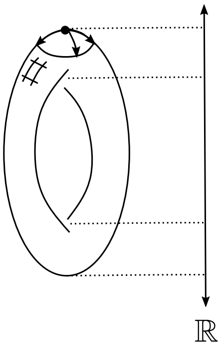
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

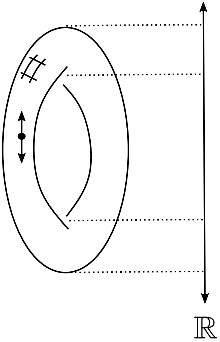
The Torus



- Critical points
- Index of a critical point

Smooth Morse Theory

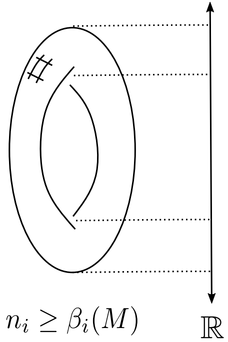
The Torus



- Critical points
- Index of a critical point
- Gradient vector fields

Smooth Morse Theory

The Torus



- Critical points
- Index of a critical point
- Gradient vector fields
- Morse inequalities

Discrete Morse Theory: Three Flavors

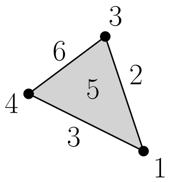
Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

and

$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$



Discrete Morse Theory: Three Flavors

Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

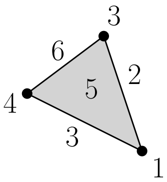
and

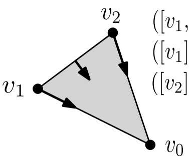
$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$

Topologically

A gradient vector field

$\{(\sigma, \tau) : \sigma \prec \tau, f(\sigma) \geq f(\tau)\}.$





$([v_1, v_2], [v_0, v_1, v_2])$
$([v_1], [v_0, v_1])$
$([v_2], [v_0, v_2])$

Discrete Morse Theory: Three Flavors

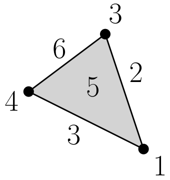
Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

and

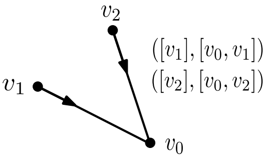
$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$



Topologically

A gradient vector field

$\{(\sigma, \tau) : \sigma \prec \tau, f(\sigma) \geq f(\tau)\}.$



Discrete Morse Theory: Three Flavors

Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

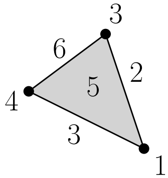
and

$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$

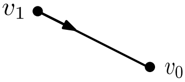
Topologically

A gradient vector field

$\{(\sigma, \tau) : \sigma \prec \tau, f(\sigma) \geq f(\tau)\}.$



$([v_1], [v_0, v_1])$



Discrete Morse Theory: Three Flavors

Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

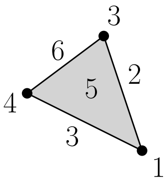
and

$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$

Topologically

A gradient vector field

$\{(\sigma, \tau) : \sigma \prec \tau, f(\sigma) \geq f(\tau)\}.$



Discrete Morse Theory: Three Flavors

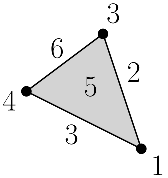
Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

and

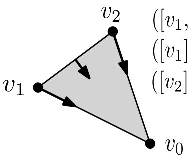
$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$



Topologically

A gradient vector field

$\{(\sigma, \tau) : \sigma \prec \tau, f(\sigma) \geq f(\tau)\}.$



$([v_1, v_2], [v_0, v_1, v_2])$
$([v_1], [v_0, v_1])$
$([v_2], [v_0, v_2])$

Discrete Morse Theory: Three Flavors

Algebraically

A function $f : K \to \mathbb{R}$ such that:

$|\{\beta \succ \sigma | f(\beta) \leq f(\sigma)\}| \leq 1,$

and

$|\{\gamma \prec \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1.$

Topologically

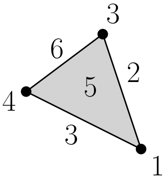
A gradient vector field

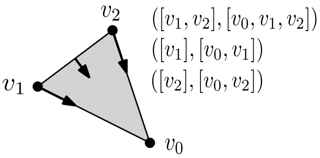
$\{(\sigma, \tau) : \sigma \prec \tau, f(\sigma) \geq f(\tau)\}.$

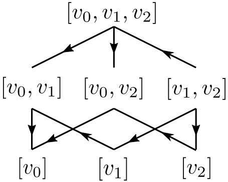
Combinatorially

A matching in the Hasse diagram

(T, H, C, m) where $m : T \to H$.



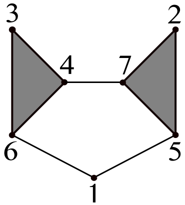


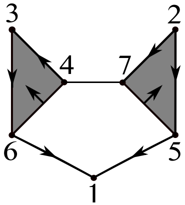


Problem

Simplicial complexes generated from point data have function values assigned to the vertices. How does one construct a discrete Morse function that:

- minimizes the number of critical simplicies
- agrees with the data?

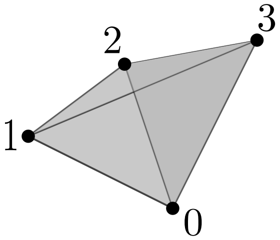




King, Knudson and Mramor

Extract

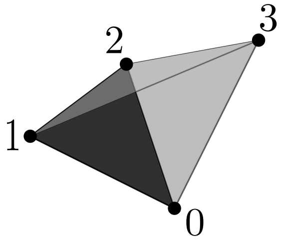
- Input: simplicial complex K, injective function $f_0 : K_0 \to \mathbb{R}$
- Output: A discrete Morse function $f : K \to \mathbb{R}$, with $f|_{K_0} = f_0$
- ExtractRaw and ExtractCancel
- Recursive call on $\mathrm{link}_K(v) := \overline{\mathrm{star}}_K(v) \setminus \mathrm{star}_K(v)$



King, Knudson and Mramor

Extract

- Input: simplicial complex K, injective function $f_0 : K_0 \to \mathbb{R}$
- Output: A discrete Morse function $f : K \to \mathbb{R}$, with $f|_{K_0} = f_0$
- ExtractRaw and ExtractCancel
- Recursive call on $\mathrm{link}_K(v) := \overline{\mathrm{star}}_K(v) \setminus \mathrm{star}_K(v)$



Properties of ExtractRaw

Unique GVF

- (H, T, C, m) consistent with f_0

Properties of ExtractRaw

Unique GVF

- (H, T, C, m) consistent with f_0
- Arrow matchings occur based off of lexicographical orderings

Properties of ExtractRaw

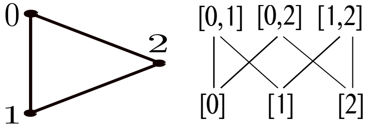
Unique GVF

- (H, T, C, m) consistent with f_0
- Arrow matchings occur based off of lexicographical orderings
- Smallest "leftmost" parent matched to largest "rightmost" child

Properties of ExtractRaw

Unique GVF

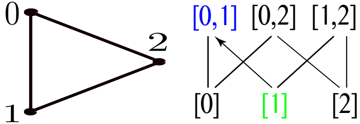
- (H, T, C, m) consistent with f_0
- Arrow matchings occur based off of lexicographical orderings
- Smallest "leftmost" parent matched to largest "rightmost" child



Properties of ExtractRaw

Unique GVF

- (H, T, C, m) consistent with f_0
- Arrow matchings occur based off of lexicographical orderings
- Smallest "leftmost" parent matched to largest "rightmost" child

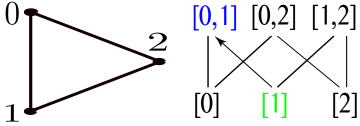


- Restrictive Output!

Properties of ExtractRaw

Unique GVF

- (H, T, C, m) consistent with f_0
- Arrow matchings occur based off of lexicographical orderings
- Smallest "leftmost" parent matched to largest "rightmost" child



- Restrictive Output!

ExtractRightChild

Hasse Decoration

- f_0 value of largest component vertex
- $\rho(\sigma)$, its rightmost child
- $l(\sigma)$, its leftmost parent

ExtractRightChild

Hasse Decoration

- f_0 value of largest component vertex
- $\rho(\sigma)$, its rightmost child
- $l(\sigma)$, its leftmost parent

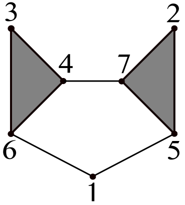
Matchings

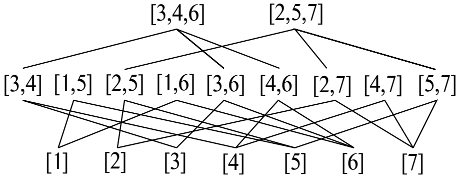
- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical

ExtractRightChild

Matchings

- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical

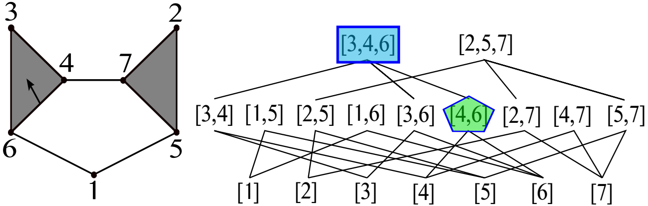




ExtractRightChild

Matchings

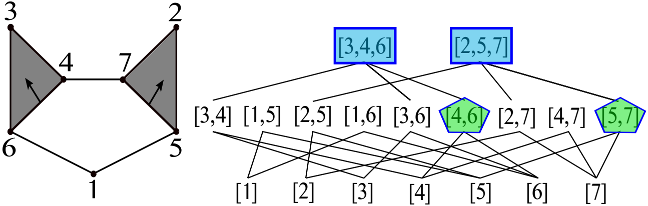
- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical



ExtractRightChild

Matchings

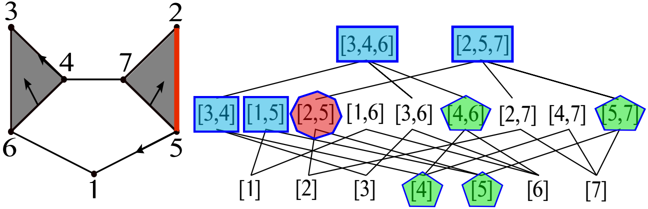
- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical



ExtractRightChild

Matchings

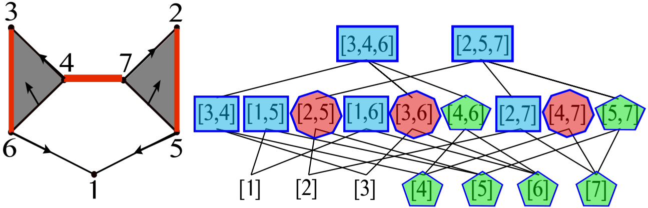
- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical



ExtractRightChild

Matchings

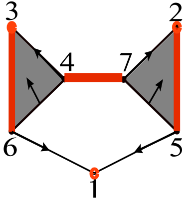
- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical

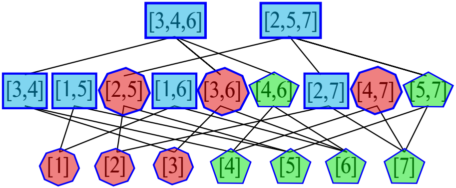


ExtractRightChild

Matchings

- Smallest lexicographical parent on Hasse diagram is head
- Largest lexicographical child on Hasse diagram is tail
- Unmatched simplices are critical





Asymptotic Analysis

Analysis of EXTRACTRIGHTCHILD

- Runs in $O(dn)$ time
- Uses $O(n)$ space

Asymptotic Analysis

Analysis of ExtractRightChild

- Runs in $O(dn)$ time
- Uses $O(n)$ space

Analysis of ExtractRaw

- Runtime lower bounded by $\Omega(n^2 logn)$
- Space complexity unknown

Summary

- Given pre-existing point data on a simplicial complex

Summary

- Given pre-existing point data on a simplicial complex
- Want a function (GVF) to inform us on its topology

Summary

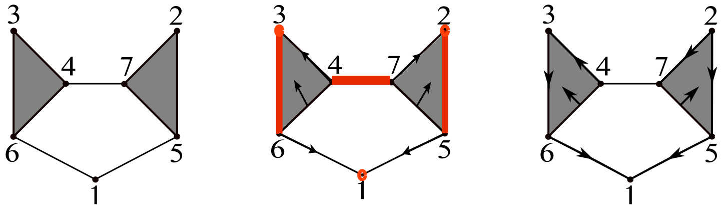
- Given pre-existing point data on a simplicial complex
- Want a function (GVF) to inform us on its topology
- Previous algorithms have an underlying property based on lexicographical orderings in the Hasse diagram

Summary

- Given pre-existing point data on a simplicial complex
- Want a function (GVF) to inform us on its topology
- Previous algorithms have an underlying property based on lexicographical orderings in the Hasse diagram
- Exploiting this property provides time complexity improvements

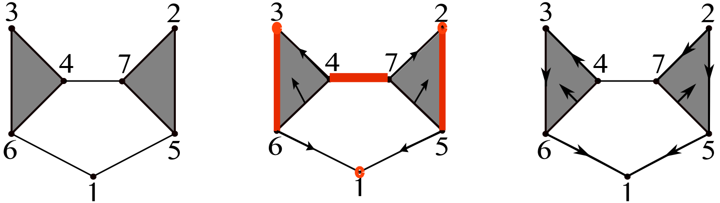
Summary

- Given pre-existing point data on a simplicial complex
- Want a function (GVF) to inform us on its topology
- Previous algorithms have an underlying property based on lexicographical orderings in the Hasse diagram
- Exploiting this property provides time complexity improvements



Summary

- Given pre-existing point data on a simplicial complex
- Want a function (GVF) to inform us on its topology
- Previous algorithms have an underlying property based on lexicographical orderings in the Hasse diagram
- Exploiting this property provides time complexity improvements



bradley.mccoy@montana.edu
benjamin.holmgren1@student.montana.edu