A Local Homology Based Distance

Brittany Terese Fasy, Tulane University
joint work with Mahmuda Ahmed and Carola Wenk

27 March 2014

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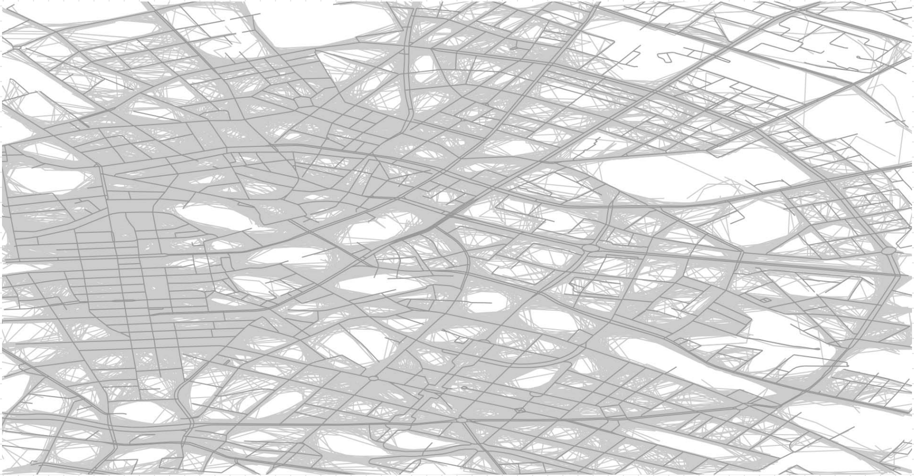

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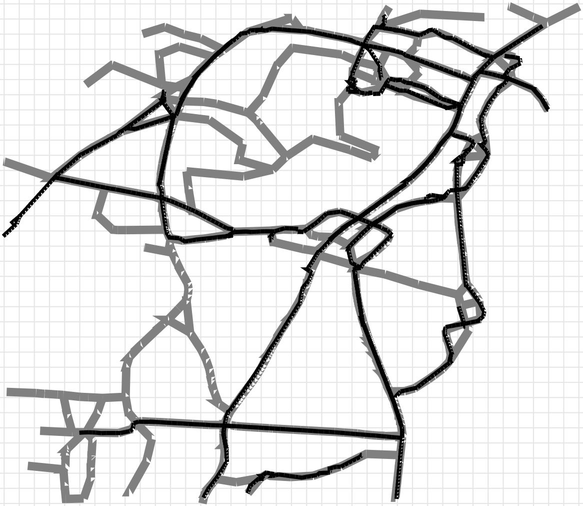

Examples

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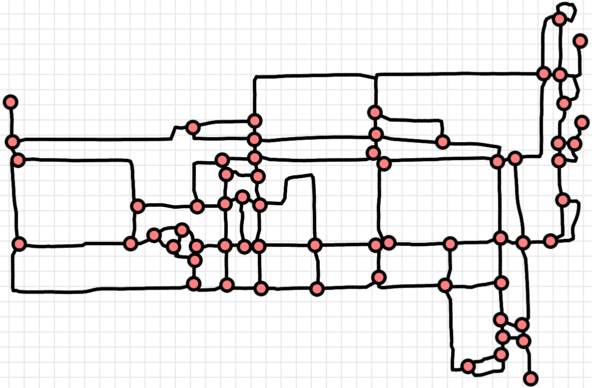
- Hiking paths: GPS paths of people.
- Migration paths: GPS on animals.
- Fillaments of galaxies: point cloud data.
- Hurricane paths: historical paths.
- **Networks: path-constrained trajectories or point sets.**

Representing Road Networks



Road Network Representation

A *road network* is represented as an embedded graph $G = (V, E) \subset D$, where $D \subset \mathbb{R}^2$ is compact.

Motivating Questions

What is the Distance Between Embedded Graphs?

Can we compare different road construction techniques?

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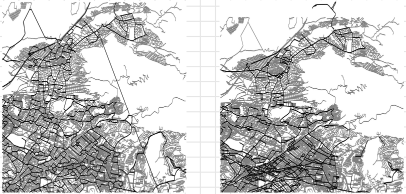
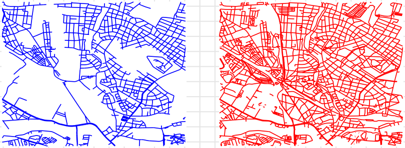


Can we detect if (and where) changes have occurred?



Different Approaches

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Hausdorff Distance

Road Networks are Sets of Points

$$G_1 = (V_1, E_1)$$

$$G_2 = (V_2, E_2)$$

Question

What is $d(G_1, G_2)$?

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Definition (Hausdorff Distance)

$$d_h^{i \to j} = \sup_{x \in G_i} \inf_{y \in G_j} ||x - y||_2$$

$$d_H(G_1, G_2) = \max(d_h^{1 \to 2}, d_h^{2 \to 1})$$

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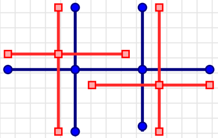

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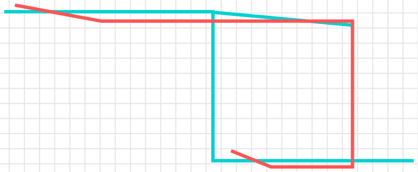
Hansel and Gretel Distance

Paths Define Neighborhood



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r is the local radius

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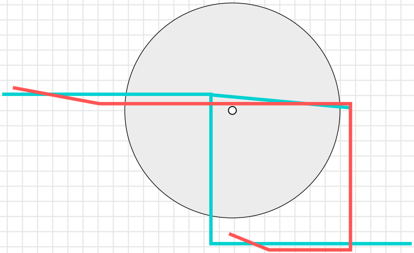

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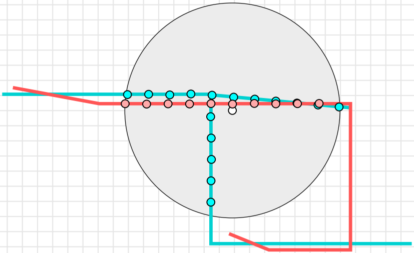

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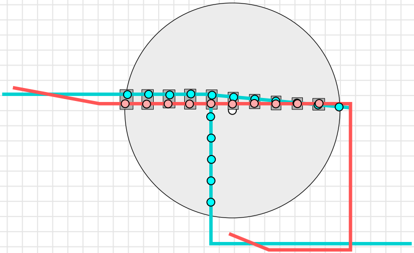

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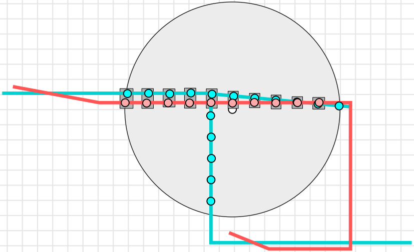
Definition (HG Distance)

$$p_{01} = k/m, \quad r_{01} = k/n$$

$$d_B(G_0, G_1) = \frac{2p_{01}r_{01}}{p_{01} + r_{01}}$$



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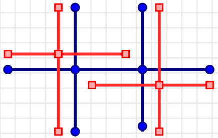

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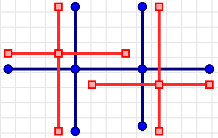

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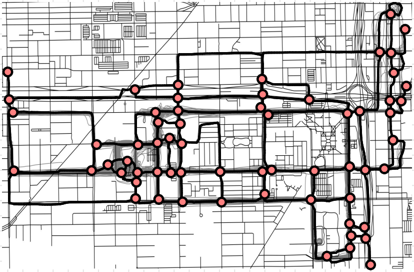

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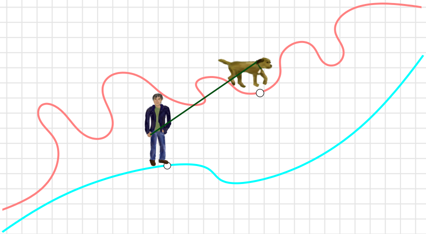
The Fréchet Distance

Definition (Fréchet Distance)

$$d_F(\alpha, \beta) = \inf_{\rho} \max_{t} \{\alpha(t), \beta(\rho(t))\}$$



Path-Based Method

Road Networks are Sets of Paths

Π_1 is the set of paths in G_1.
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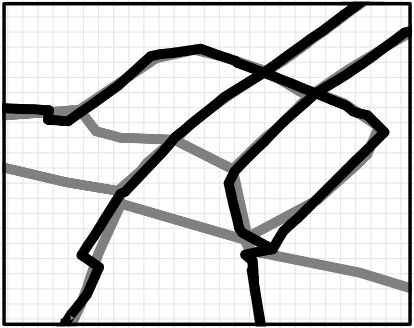

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What is $d(G_1, G_2)$?

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$$d_p(G_1, G_2) = \sup_{\alpha \in \Pi_1} \inf_{\beta \in \Pi_2} d_F(\alpha, \beta)$$

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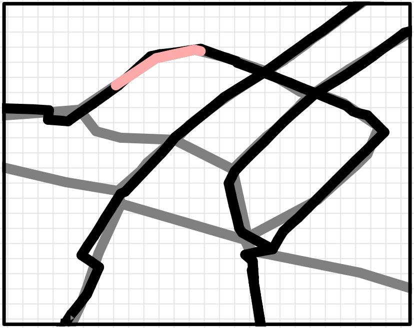

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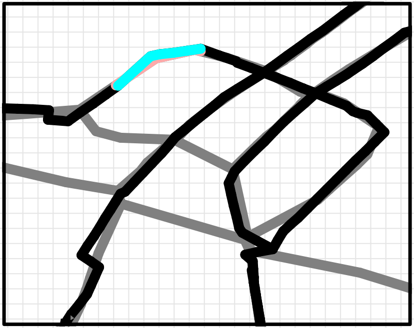

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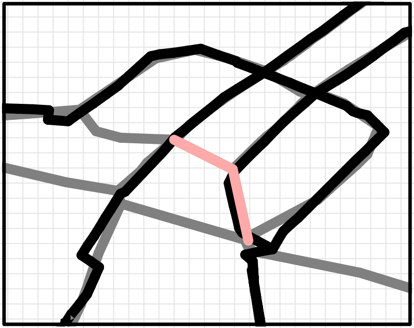

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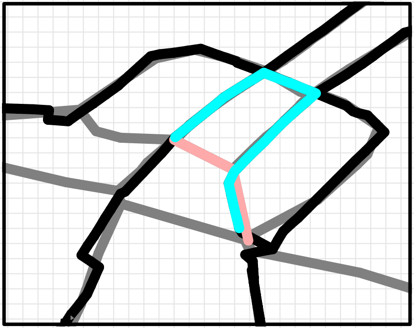

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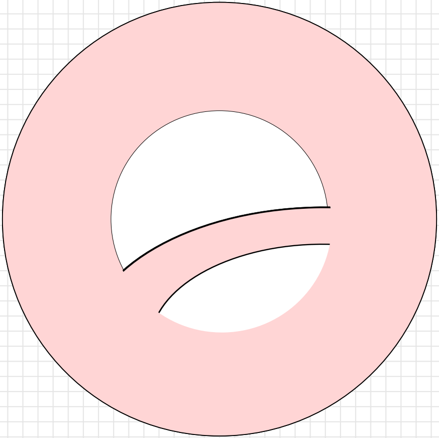
Homology and Relative Homology

Definitions by Pictures



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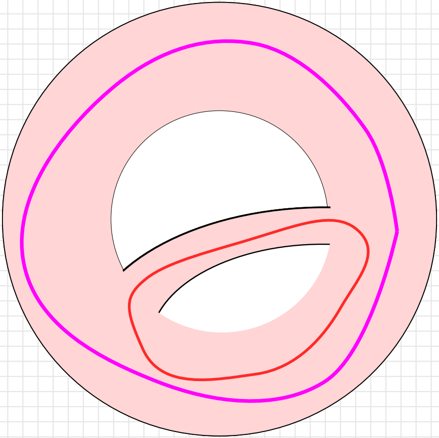

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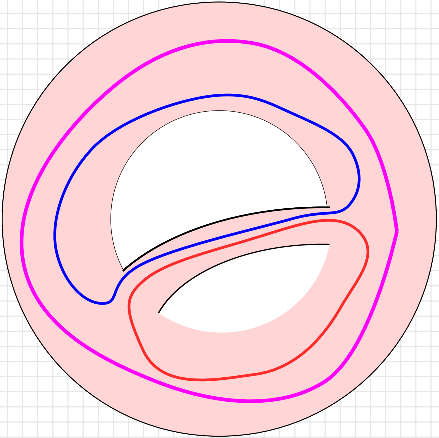

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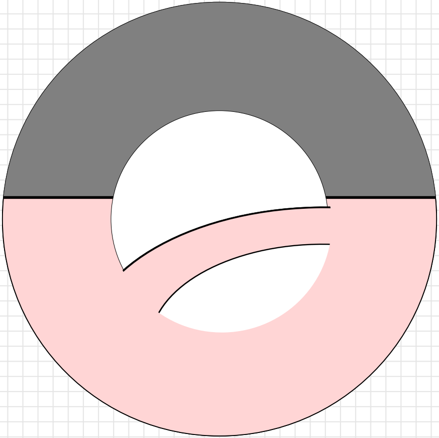

$$H_1(\mathbb{X}, A)$$

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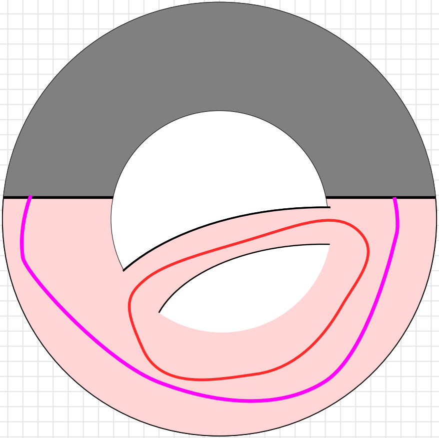

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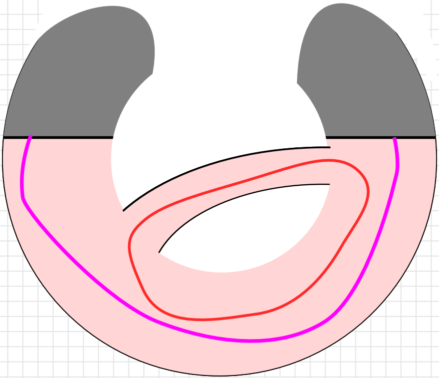

$$H_1(\mathbb{X}, A)$$

Local Homology

A Definition

Definition (Local Homology)

$$H_1(\mathbb{X}, \mathbb{X} - x) = \lim_{r \to 0} H_1(\mathbb{X} \cap B_r(x), \mathbb{X} \cap \partial B_r(x))$$

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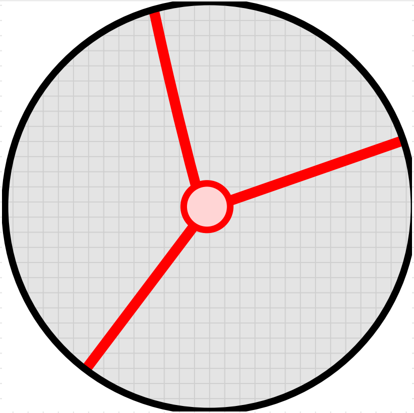

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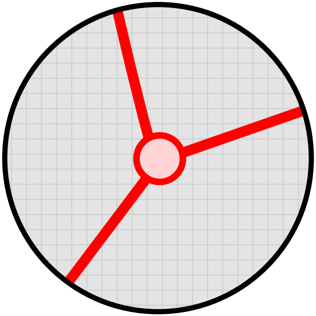

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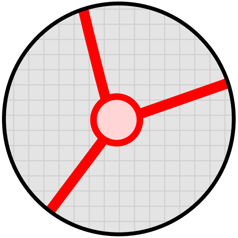

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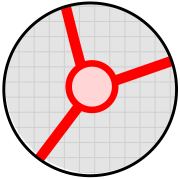

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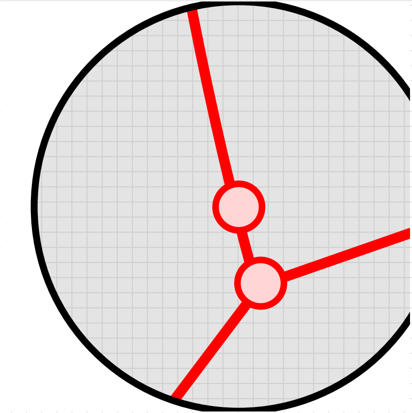

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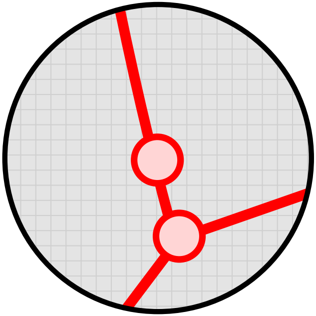

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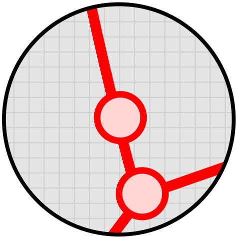

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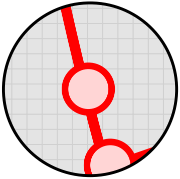

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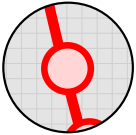

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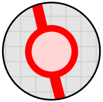

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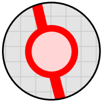
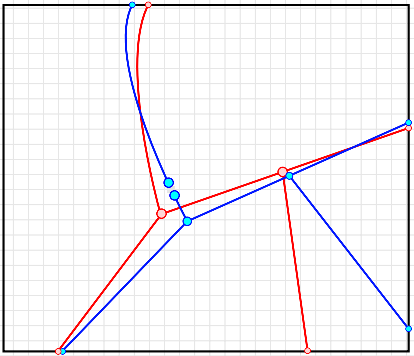


Theorem

$$H_1(\mathbb{X} \cap B_r(x), \mathbb{X} \cap \partial B_r(x))$$

$$= H_1(\mathbb{X}/\mathbb{X} - B_r(x))$$

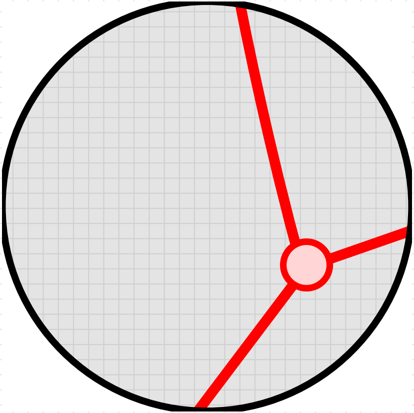
Local Topology Approach [AFW]



We will use the embedding and the local homology to create a *local distance signature*.

Local Distance Signature

Finding the Local Persistence Diagram



$G_i \subset D$ is the road network.
$r > 0$ is the scale.
$x \in D$.

$$LG_i(x, 0) = (G_i \cap B_r(x))/(G_1 \cap \partial B_r(x))$$

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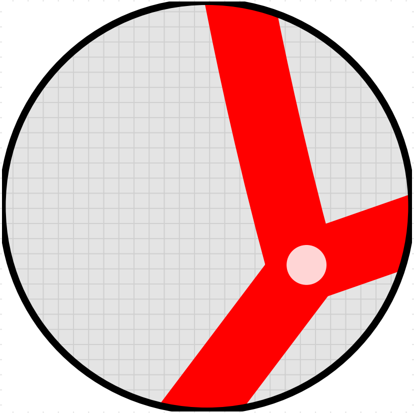

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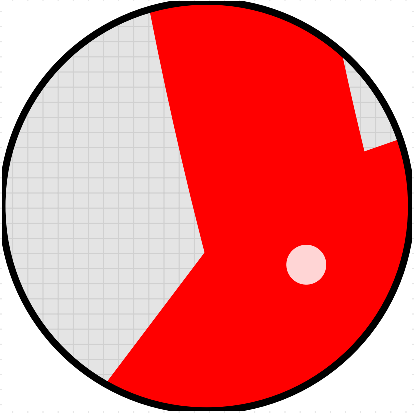
$$LG_i(x, \varepsilon) = (G_i{}^{\varepsilon} \cap B_r(x))/(G_1{}^{\varepsilon} \cap \partial B_r(x))$$

Local Distance Signature

Finding the Local Persistence Diagram

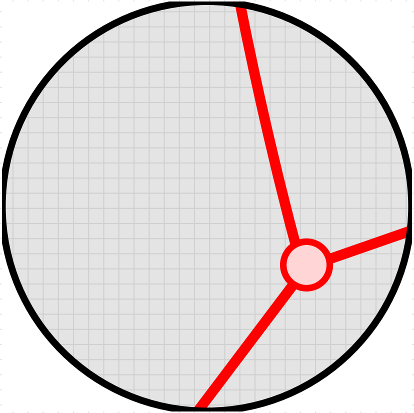


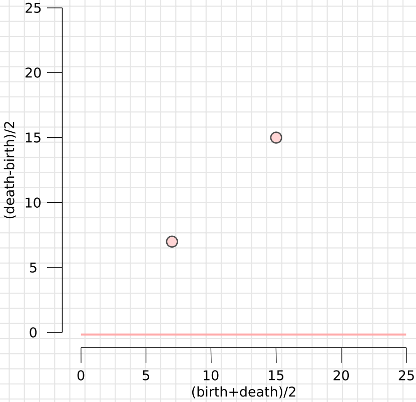
$G_i \subset D$ is the road network.
$r > 0$ is the scale.
$x \in D$.

$$LG_i(x, 2\varepsilon) = ({G_i}^{2\varepsilon} \cap B_r(x))/({G_1}^{2\varepsilon} \cap \partial B_r(x))$$

Local Distance Signature

Finding the Local Persistence Diagram



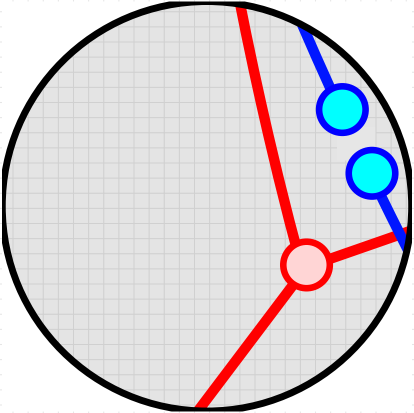


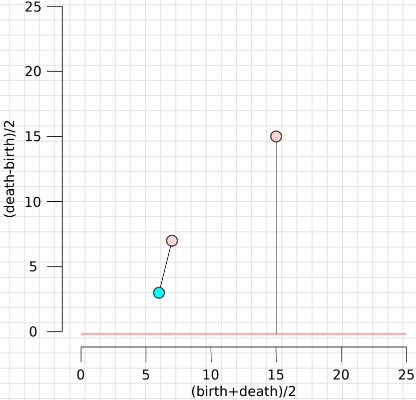
$$f_{1,x,r} \colon B_r(x)/\partial B_r(x) \to \mathbb{R}$$

$$f_{1,x,r}(y) = d(y, G_i)$$

Local Distance Signature

Computing the Local Distance





$$s(x,r) := W_\infty(\mathrm{Dgm}_1(f_{1,x,r}), \mathrm{Dgm}_1(f_{2,x,r}))$$

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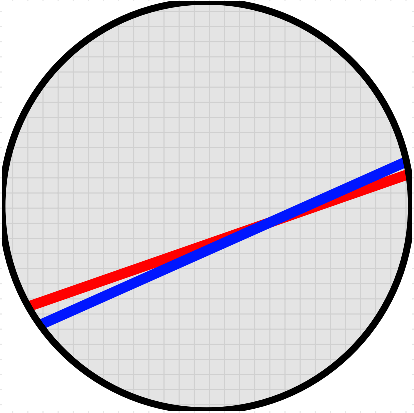
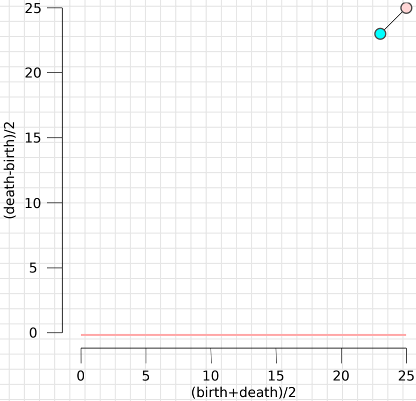
Computing the Local Distance





$$s(x, r) := W_\infty(\mathrm{Dgm}_1(f_{1,x,r}), \mathrm{Dgm}_1(f_{2,x,r}))$$

Local Topology Based Distance

Definition (Local Homology Distance)

$$d_{LH}(G_1, G_2) = \frac{1}{|D|} \int_D s(x, r)\, dx$$

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Properties

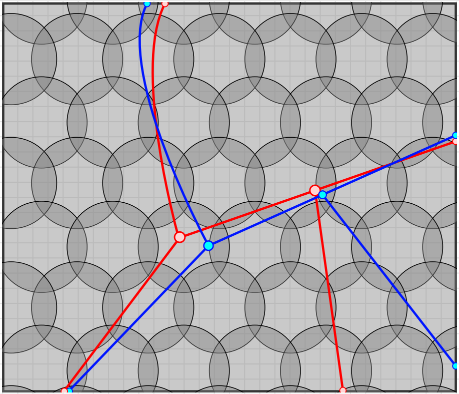
1. $r_0 = 0 \implies$ metric
2. $d_{LH}(G_1, G_2) \leq ||f_1 - f_2||_\infty$

Properties

- $s(\cdot, r)$ is 1-Lipschitz

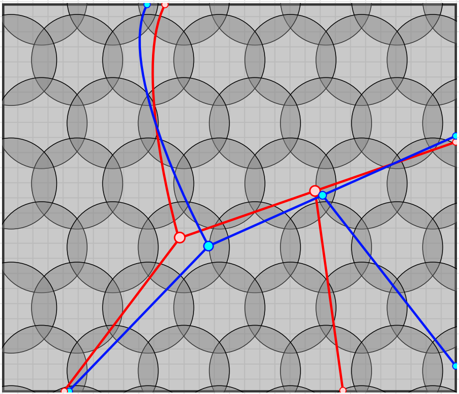
Properties

- $s(\cdot, r)$ is 1-Lipschitz $\implies$ can estimate with a finite cover



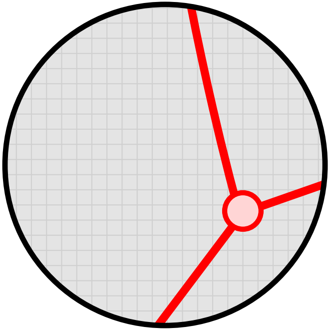
Properties

- $s(\cdot, r)$ is 1-Lipschitz $\implies$ can estimate with a finite cover
- Error bounded by (a function of) the min distance between ball centers.

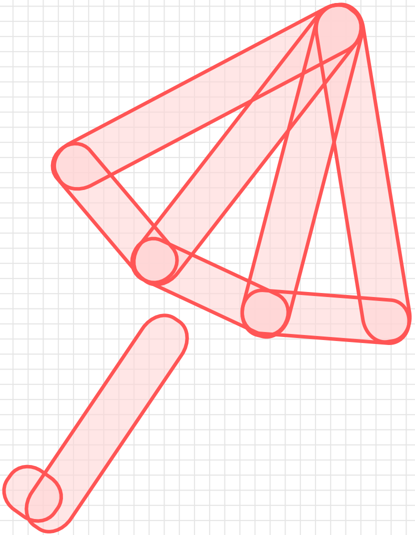


Computation

Using the Nerve



Nerve



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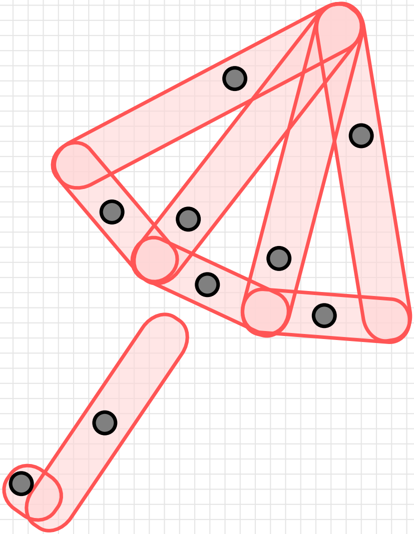

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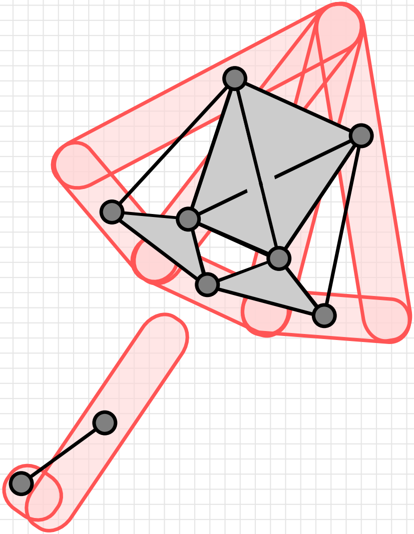

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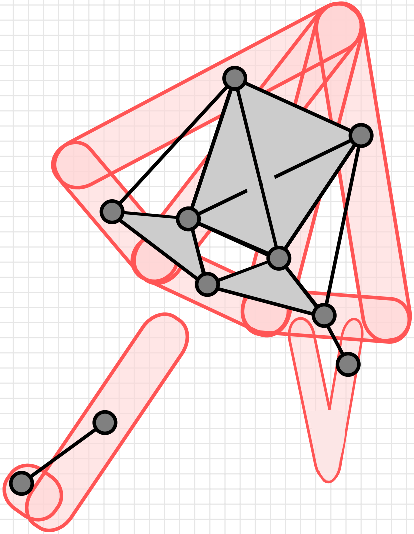

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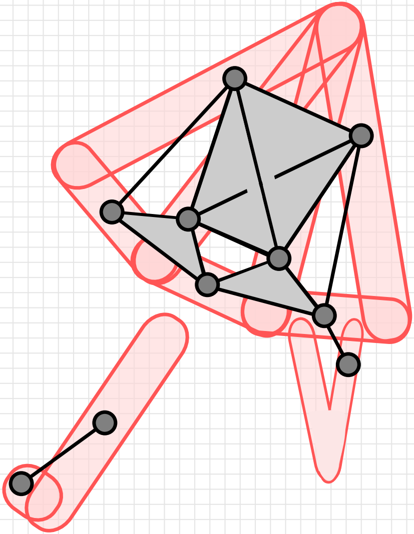
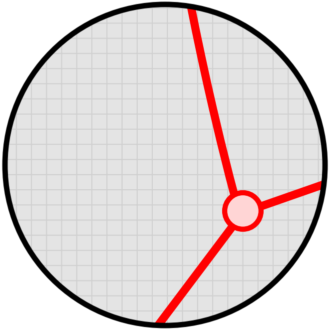


Nerve Lemma

If U is a collection of sets such that all intersections are either empty or contractible, then $H(U) = H(N(U))$.

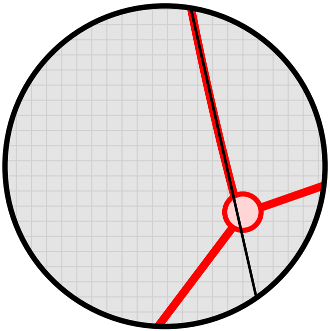
Computation

Using the Nerve



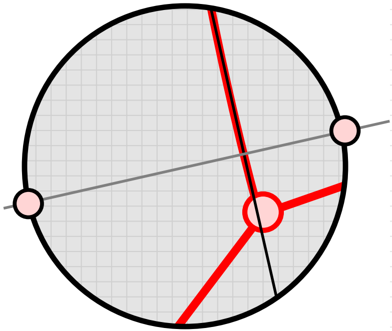
Computation

Using the Nerve



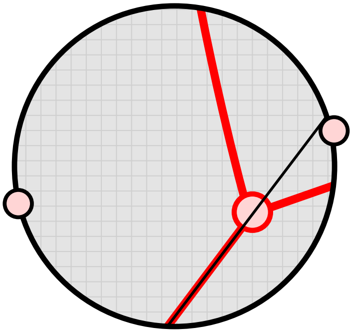
Computation

Using the Nerve



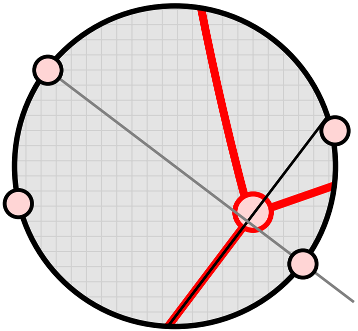
Computation

Using the Nerve



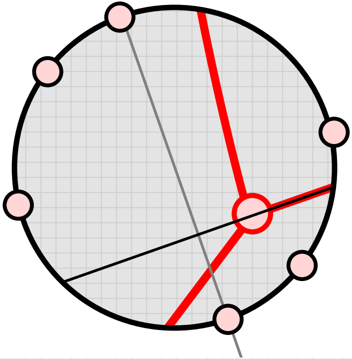
Computation

Using the Nerve



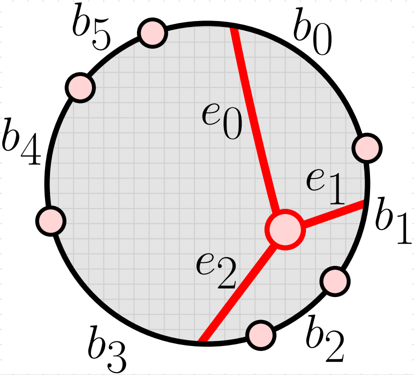
Computation

Using the Nerve



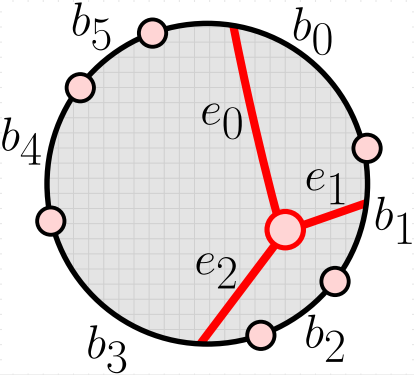
Computation

Using the Nerve



Computation

Using the Nerve

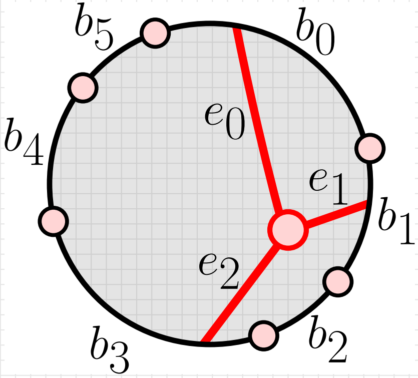


1-Skeleton

edge-edge: $d(e_i, e_j)$
boundary-boundary: $d(b_i, b_j) = 0$
boundary-edge: $d(e_i, b_j)$

Computation

Using the Nerve



1-Skeleton

edge-edge: $d(e_i, e_j)$
boundary-boundary: $d(b_i, b_j) = 0$
boundary-edge: $d(e_i, b_j)$

2-Skeleton

e-e-e: $d(e_i, e_j, e_k)$
b-b-b: $d(b_i, b_j, b_k) = 0$
b-e-e: $d(b_i, e_j, e_k)$
b-b-e: $d(b_i, b_j, e_k)$

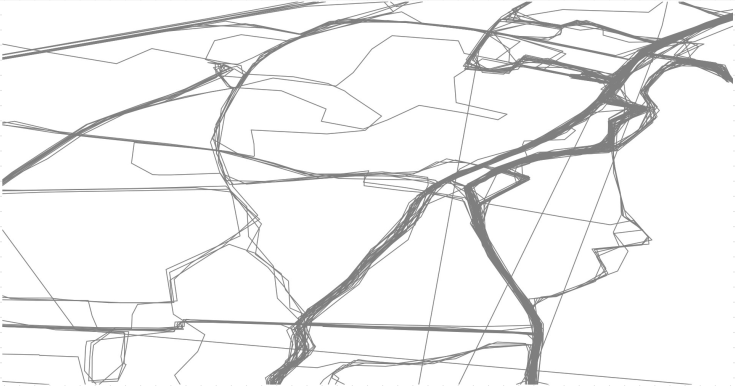
Dataset 1: Athens

www.mapconstruction.org

- 129 GPS trajectories from school buses.
- D = 2.6 km x 6 km
- Two reconstruction algorithms: [BE-12a] and [KP-12].
- Trajectories: 13-47 samples

Dataset 1: Athens

www.mapconstruction.org



Dataset 1: Athens

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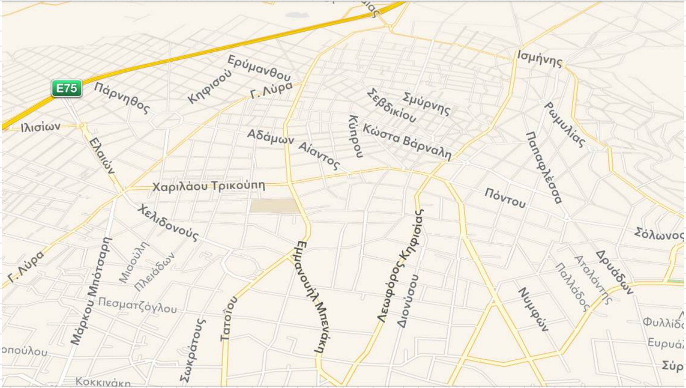

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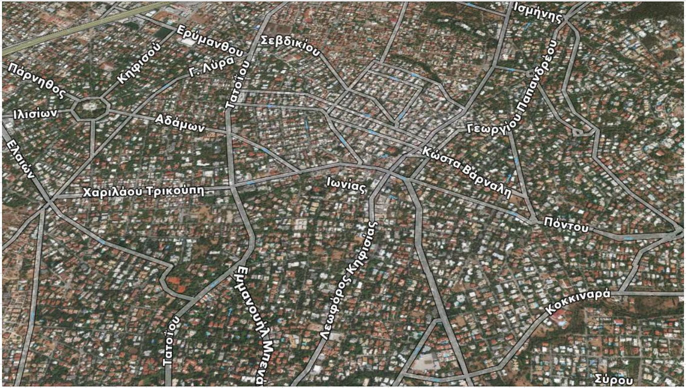
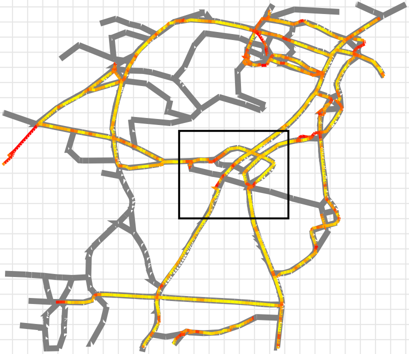


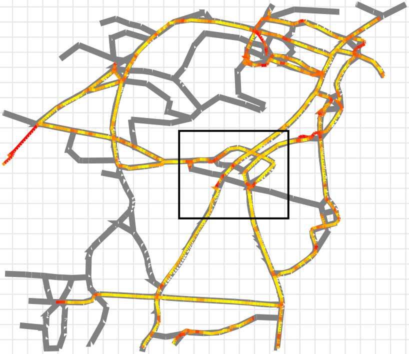
Results

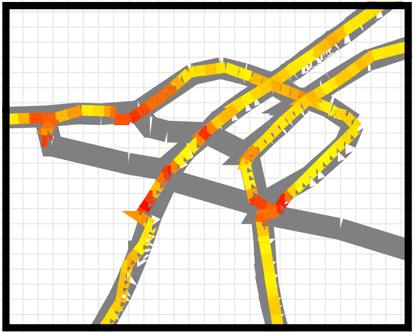
Athens: Comparing Two Different Reconstructions



Results

Athens: Comparing Two Different Reconstructions

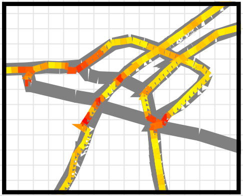




Ongoing

- Apply this to different construction algorithms versus the ground truth to rank algorithms.
- Fast Implementation of distance measure (currently, estimating the distance).
- Improve theoretical guarantees.
- Input Model: other noise models?
- Output Model: road category, direction, intersection regions, ...

Summary



Map Comparison

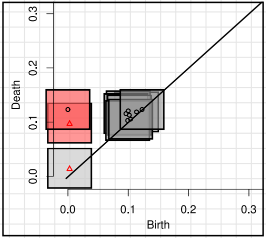
[AFW-14] Local Homology Based Distance Between Maps.

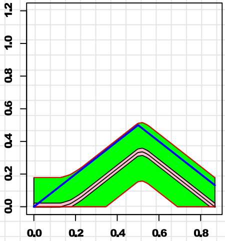
- Provide a local distance signature.
- ... that is computable.
- ... and extends to a distance metric between graphs.

On Another Note ...

Joint Work with F. Chazal, F. Lecci, A. Rinaldo, L. Wasserman

- Confidence sets for persistence diagrams: $(\hat{P}, \delta)$
- Confidence bands for functional summaries of persistence diagrams.
- Hypothesis testing in TDA.





Thank You!

Contact me: brittany.fasy@alumni.duke.edu
www.mapconstruction.org, www.fasy.us